Lecture 7: Continuity

Mathematical Analysis

1 Continuity

Definition 1.1 (Function Continuity at a Point). [def:function-continuity] Under metric space (A, d_A) and (B, d_B), let $f : A \to B$. f is continous at a point $c \in A$ if

$$\forall \varepsilon : \mathbb{R}_+, \exists \delta : \mathbb{R}_+ \, \forall x \in B_\delta(c), \, d_B(f(x), f(c)) < \varepsilon$$

Lemma 1.1. [lem:continuity-and-function-limit] *If a function f's limit exists at a point $c \in A$ which is $\ell \in B$, and f is continuous at c and $\ell = f(c)$*

Lemma 1.2. [lem:continuity-at-isolated-points] *Every isolated point c of a set A is a continous point*

Proposition 1.1. [prop:continuity-limit-or-isolated] *A point c is continous on function f if and only if either its limit exists and equals to $f(c)$ or it is an isolated point of the domain*

Definition 1.2 (Continuity on a Set). [def:continuous-on-set] A function f is continuous on a set A if for every point $c \in A$, f is continuous at c, ie

$$\forall c : A, \forall \varepsilon : \mathbb{R}_+, \exists \delta : \mathbb{R}_+ \, \forall x \in B_\delta(c), \, d_B(f(x), f(c)) < \varepsilon$$

Remark 1.1. The continuity of a function is a local property, which means that it only depends on the behavior of the function in an arbitrarily small neighborhood around each point in its domain.
When we say a function is continuous on a set/domain, we mean that it is continuous at every point in that set/domain.

2 Properties of Continuous Points

Corollary 2.1 (Equivalent Criteria for Continuity). [cor:continuity-epsilon-delta] *Let (A, d_A) and (B, d_B) be metric spaces, let $f : A \to B$, and let $c \in A$. Each of the following conditions is equivalent to f being continuous at c*

1. *The distance condition*

$$\begin{gathered} \forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall x \in A, \\ d_A(c, x) < \delta \implies d_B(f(c), f(x)) < \varepsilon \end{gathered}$$

2. *The ball condition, with input balls taken in A and output balls in B*

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall x \in B_\delta(c), \quad f(x) \in B_\varepsilon(f(c))$$

Theorem 2.1. [thm:sequential-criterion-for-continuity] *f is continous at a point c if and only if for a convergent sequence a having a codomain A*

$$f(\lim(a)) = \lim n \mapsto f(a(n))$$

Corollary 2.2. [cor:sequence-test-for-discontinuity] *f is not continous at c If there exists a sequence a having a codomain A such that*

$$\lim a = c, \lim n \mapsto f(a(n)) \neq f(c)$$

Theorem 2.2. [thm:arithmetic-of-continuous-functions] *If f, g both continous at a point c, then*

1. *$f + g$ is continous at c*
2. *kf is continous at c for any constant k*
3. *$f \cdot g$ is continous at c*
4. *$\frac{f}{g}$ is continous at c if $g(c) \neq 0$*

Proof. **TODO**Move limit out and use properties of seqlim □

Theorem 2.3. [thm:continuity-of-composition] *If f, g both continous at a point c, then $f \circ g$ is continous at c*

3 Continous Functions on Compact Sets

Theorem 3.1 (Continuous Images of Compact Sets)**.** [thm:compact-image] *Let (X, d_X) and (B, d_B) be metric spaces. If $f : X \to B$ is a continuous function on a compact set $A \subseteq X$, then $f^{\text{img}}(A)$ is compact*

Proof. Define any sequence $y : \mathbb{N} \to f^{\text{img}}(A)$
Accordig to that we can define another sequence $x : \mathbb{N} \to A$ such that $x \equiv n \mapsto f^{-1}(y(n))$ where f^{-1} is the inverse function of f restricted to $f^{\text{img}}(A)$. In other words, $\forall n \in \mathbb{N}_1$, $f(x(n)) = y(n)$
By sequential compactness of A, take a subsequence $k \mapsto x(n(k))$ and a point $a \in A$ such that $x(n(k)) \to a$, where $n : \mathbb{N}_1 \to \mathbb{N}_1$ is strictly increasing
Given $\varepsilon \in \mathbb{R}_+$, continuity at a gives $\delta \in \mathbb{R}_+$ such that for every $u \in A$,

$$d_X(u, a) < \delta \implies d_B(f(u), f(a)) < \varepsilon$$

Since $x(n(k)) \to a$, there exists $N \in \mathbb{N}_1$ such that for every $k \in \mathbb{N}_1$ with $k \geq N$, $d_X(x(n(k)), a) < \delta$. Hence

$$d_B(y(n(k)), f(a)) = d_B(f(x(n(k))), f(a)) < \varepsilon$$

Thus $y(n(k)) \to f(a) \in f^{\text{img}}(A)$. Every sequence in $f^{\text{img}}(A)$ therefore has a subsequence converging to a point of $f^{\text{img}}(A)$, so sequential compactness shows that $f^{\text{img}}(A)$ is compact □

Theorem 3.2 (Extreme Value Theorem)**.** [thm:extreme-value-theorem] *If $K \subseteq \mathbb{R}$ is nonempty and compact and $f : K \to \mathbb{R}$ is continuous, then there exist points $x_m, x_M \in K$ such that*

$$f(x_m) = \min(f^{\text{img}}(K)) \quad \textit{and} \quad f(x_M) = \max(f^{\text{img}}(K))$$

Proof. Put $S \equiv f^{\text{img}}(K)$. By compactness of continuous images, S is nonempty and compact. By attainment of bounds on compact sets,

$$m \equiv \inf(S) = \min(S) \in S, \qquad M \equiv \sup(S) = \max(S) \in S$$

By the definition of image,

$$\exists x_m, x_M \in K, \quad f(x_m) = m, \quad f(x_M) = M$$

□

4 Uniform Continuity

Definition 4.1 (Uniform Continuity)**.** [def:uniform-continuity] Let (X, d_X) and (B, d_B) be metric spaces, let $A \subseteq X$, and let $f : X \to B$. The function f is *uniformly continuous* on A if and only if

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall x, y \in A,$$
$$d_X(x, y) < \delta \implies d_B(f(x), f(y)) < \varepsilon$$

Theorem 4.1 (Uniform Continuity Implies Continuity). [thm:uniform-continuity-implies-continuity] *If f is uniformly continuous on A, then its restriction to A is continuous*

Remark 4.1 (Dependence of δ). [rem:uniform-continuity-delta-dependence] For continuity on a set, δ may depend on both ε and the point c: the point is fixed before δ is chosen
For uniform continuity, δ may depend on ε, but must be chosen before the points. **For each fixed $\varepsilon > 0$, the same $\delta > 0$ must work for every pair of points in A**

Remark 4.2 (Board Comparison). [rem:continuity-uniform-comparison] For $f : X \to B$ and $A \subseteq X$, compare continuity of the restriction to A with uniform continuity on A

	Continuity	**Uniform Continuity**
Space	(X, d_X) (B, d_B)	(X, d_X) (B, d_B)
Quantifier order	$\forall c \in A,\ \forall \varepsilon \in \mathbb{R}_+$ $\exists \delta \in \mathbb{R}_+,\ \forall x \in A$	$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in \mathbb{R}_+$ $\forall c, x \in A$
Allowed dependence	$\delta = \delta(\varepsilon, c)$	$\delta = \delta(\varepsilon)$
Input restriction	$d_X(x, c) < \delta$	$d_X(x, c) < \delta$
Output requirement	$d_B(f(x), f(c)) < \varepsilon$	$d_B(f(x), f(c)) < \varepsilon$

Example 4.1 (Reciprocal Function). [eg:reciprocal-not-uniformly-continuous] The function

$$f : (0, 1) \to \mathbb{R} \equiv x \mapsto \frac{1}{x}$$

is continuous, but not uniformly continuous. Fix $\varepsilon = 1$. For any $\delta > 0$, choose $0 < t < \min(\delta, 1/2)$ and put $x = t$, $y = 2 \cdot t$. Then $x, y \in (0, 1)$, but

$$|x - y| = t < \delta, \qquad |f(x) - f(y)| = \frac{1}{2 \cdot t} > 1$$

Thus no positive δ works for every pair of points when $\varepsilon = 1$. At each fixed point, continuity still allows a positive radius depending on that point and the tolerance

5 Intermediate Value Theorem

Theorem 5.1 (Intermediate Value Theorem). [thm:ivt] *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous. For every $L \in \mathbb{R}$,*

$$\big(f(a) < L < f(b) \text{ or } f(b) < L < f(a)\big) \implies \exists c \in (a, b), \quad f(c) = L$$

Proof. **TODO** □

Corollary 5.1 (Fixed Point Theorem for an Interval). [cor:interval-fixed-point] *Let $a, b \in \mathbb{R}$ with $a \leq b$. If $f : [a, b] \to [a, b]$ is continuous, then*

$$\exists c \in [a, b], \quad f(c) = c$$

Proof. Define $g : [a, b] \to \mathbb{R} \equiv x \mapsto f(x) - x$. Since $x \mapsto x$ is a continuous function on the closed interval $[a, b]$, then according to arithmetic of continuous functions, g is also continuous on $[a, b]$
We want to prove that $\exists c \in [a, b]$, $g(c) = 0$ since $g(c) = 0 \iff f(c) = c$
Plug in a and b into g,

$$g(a) = f(a) - a, \quad g(b) = f(b) - b$$

Notice that

$$\forall x \in [a, b], \quad f(x) \in [a, b]$$

Then, $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$
If $g(a) = 0$ or $g(b) = 0$, the corresponding endpoint is a fixed point. Otherwise, $a < b$ and $g(a) > 0 > g(b)$. According to the Intermediate Value Theorem, there exists $c \in (a, b)$ such that $g(c) = 0$, so the fixed point exists □