Lecture 6: Limit of Functions and Proving Limits Systematically

Mathematical Analysis

1 Limit of Functions

Definition 1.1 (Existence of a Limit at a Point). [def:limit-of-functions] Let (X, d_X) and (B, d_B) be metric spaces, let $A \subseteq X$ and $f : A \to B$, and let $c \in X$ be a limit point of A
If there exists a point $\ell \in B$ such that

$$\forall \varepsilon \in \mathbb{R}_+,\, \exists \delta \in \mathbb{R}_+,\, \forall x \in A,\, 0 < d_X(x, c) < \delta \implies d_B(f(x), \ell) < \varepsilon$$

then we say that f has a limit at c

Theorem 1.1 (Uniqueness of a Function Limit). [thm:function-limit-uniqueness] *With the setup of the existence definition, if a function f has a limit at a point c, then the limit is unique, ie*

$$\exists \ell : B,\, \forall \varepsilon \in \mathbb{R}_+,\, \exists \delta \in \mathbb{R}_+,\, \forall x \in A,\, 0 < d_X(x, c) < \delta \implies d_B(f(x), \ell) < \varepsilon$$

$$\iff$$

$$\exists! \ell : B,\, \forall \varepsilon \in \mathbb{R}_+,\, \exists \delta \in \mathbb{R}_+,\, \forall x \in A,\, 0 < d_X(x, c) < \delta \implies d_B(f(x), \ell) < \varepsilon$$

Proof. Assume that

$$\exists \ell_1,\, \forall \varepsilon \in \mathbb{R}_+,\, \exists \delta_1 \in \mathbb{R}_+,\, \forall x \in A,\, 0 < d_X(x, c) < \delta_1 \implies d_B(f(x), \ell_1) < \varepsilon$$

$$\exists \ell_2,\, \forall \varepsilon \in \mathbb{R}_+,\, \exists \delta_2 \in \mathbb{R}_+,\, \forall x \in A,\, 0 < d_X(x, c) < \delta_2 \implies d_B(f(x), \ell_2) < \varepsilon$$

where $\ell_1 \neq \ell_2$
Fix $\varepsilon > 0$ and choose the corresponding δ_1, δ_2. Since c is a limit point of A, choose $x \in A \cap B_{\min(\delta_1,\delta_2)}(c) \setminus \{c\}$
We know that

$$\forall x \in A \cap B_{\min(\delta_1,\delta_2)}(c) \setminus \{c\},\, d_B(f(x), \ell_1) + d_B(f(x), \ell_2) < 2\varepsilon$$

Then, according to triangle inequality,

$$d_B(\ell_1, \ell_2) \leq d_B(f(x), \ell_1) + d_B(f(x), \ell_2) < 2\varepsilon$$

According to epsilon criteria,

$$d_B(\ell_1, \ell_2) = 0$$

ie $\ell_1 = \ell_2$. Contradiction. So uniqueness is proven. □

Remark 1.1 (Board Comparison). [rem:sequence-function-limit-challenges] For a sequence $a : \mathbb{N}_1 \to X$ in a metric space (X, d) and a function $f : A \to B$, compare the two challenges

	Sequence	**Function**
Arbitrary tolerance	$\forall \varepsilon \in \mathbb{R}_+$	$\forall \varepsilon \in \mathbb{R}_+$
Restrictor	$\exists N \in \mathbb{N}_1$	$\exists \delta \in \mathbb{R}_+$
Input restriction	$n \geq N$	$0 < d_X(x, c) < \delta$
Output requirement	$d(a(n), \ell) < \varepsilon$	$d_B(f(x), \ell) < \varepsilon$

The output requirement must hold for every $n \in \mathbb{N}_1$ or $x \in A$ satisfying its input restriction

1. ε is arbitrary small
2. δ and N is used to restrict x st x is close enough to the limit
3. δ and N are all depends on ε

Convention 1.1 (The ε–δ Challenge)**.** [conv:epsilon-delta-challenge] A value $\ell \in B$ passes the ε–δ *challenge* for f at c if, for any proposed $\varepsilon > 0$, there exists a radius $\delta > 0$ such that for all $x \in A$ with $0 < d_X(x, c) < \delta$, $d_B(f(x), \ell) < \varepsilon$

Theorem 1.2 (Function Limit Challenge Replacement)**.** [thm:function-limit-challenge-replacement] *Let (X, d_X) and (B, d_B) be metric spaces, let $A \subseteq X$ and $f : A \to B$, let $c \in X$ be a limit point of A, and fix $\ell \in B$. Let g be a convergence challenge function. With balls taken in (X, d_X),*

$$\big(\forall \varepsilon \in \mathbb{R}_+,\, \exists \delta \in \mathbb{R}_+,\, \forall x \in A \cap B_\delta(c) \setminus \{c\}, \quad d_B(f(x), \ell) < \varepsilon\big)$$
$$\iff$$
$$\big(\forall \varepsilon \in \mathbb{R}_+,\, \exists \delta \in \mathbb{R}_+,\, \forall x \in A \cap B_\delta(c) \setminus \{c\}, \quad d_B(f(x), \ell) < g(\varepsilon)\big)$$

Proof. Similar to the sequence challenge-replacement proof □

2 Function Limit Operator

Definition 2.1 (Limit of a Function at a Point)**.** [def:function-limit-relation] Let (X, d_X) and (B, d_B) be metric spaces, let $A \subseteq X$ and $f : A \to B$, and let $c \in X$ be a limit point of A. We say that *f has limit $\ell \in B$ at c* if

$$\begin{gathered} \forall \varepsilon \in \mathbb{R}_+, \quad \exists \delta \in \mathbb{R}_+, \quad \forall x \in A, \\ 0 < d_X(x, c) < \delta \implies d_B(f(x), \ell) < \varepsilon \end{gathered}$$

Definition 2.2 (Limit Domain)**.** [def:function-limit-domain] A point in the limit domain if and only if

1. c is a limit point of the domain A
2. f has a limit at c

$$\operatorname{LimDom} : (A \to B) \to \mathcal{P}(X)$$
$$\operatorname{LimDom}(f) \equiv \left\{ c \in X \;\middle|\; \begin{gathered} \forall r \in \mathbb{R}_+, \quad B_r(c) \cap \big(A \setminus \{c\}\big) \neq \emptyset \\ \wedge \\ \exists \ell \in B, \quad \forall \varepsilon \in \mathbb{R}_+, \quad \exists \delta \in \mathbb{R}_+, \\ \forall x \in A, \quad 0 < d_X(x, c) < \delta \implies d_B(f(x), \ell) < \varepsilon \end{gathered} \right\}$$

`LaTeX: \LimDom (f)`

Proposition 2.1 (Well-Definedness of the Function Limit)**.** [prop:function-limit-well-defined] *With the setup of the limit relation, for every $f : A \to B$ and $c \in \operatorname{LimDom}(f)$, there exists a unique $\ell \in B$ such that f has limit ℓ at c*

Proof. Existence follows from the definition of the limit domain. The limit is unique by the uniqueness theorem □

Definition 2.3 (Function Limit Operator)**.** [def:function-limit-op]

$$\lim : \Pi f : (A \to B), \quad \operatorname{LimDom}(f) \to B$$
$$\lim(f)(c) = \ell \iff \begin{gathered} \forall \varepsilon \in \mathbb{R}_+, \quad \exists \delta \in \mathbb{R}_+, \quad \forall x \in A, \\ 0 < d_X(x, c) < \delta \implies d_B(f(x), \ell) < \varepsilon \end{gathered}$$

`LaTeX: \fnlim (f)(c)`

Convention 2.1 (Function Limit Notation). [conv:function-limit-notation] For the function limit operator, write

$$\lim_c(f) \equiv \lim(f)(c)$$

LaTeX: \fnlim _c(f)

3 Properties of Function Limits

Theorem 3.1. *Let (X, d_X) and (B, d_B) be metric spaces, let $A \subseteq X$, $f : A \to B$, $c \in \text{LimDom}(f)$, and $\ell \in B$. For every sequence $a : \mathbb{N}_1 \to A \setminus \{c\}$ converging to c in X, the image sequence $n \mapsto f(a(n))$ converges in B, and*

$$\lim_c(f) = \ell \iff \forall a \text{ as above}, \quad \lim(n \mapsto f(a(n))) = \ell$$

Proof. **TODO** □

Corollary 3.1. *Let (X, d_X) and (B, d_B) be metric spaces, let $A \subseteq X$ and $f : A \to B$, and let $c \in X$ be a limit point of A. Let $a_1, a_2 : \mathbb{N}_1 \to A \setminus \{c\}$. Assume both image sequences $n \mapsto f(a_1(n))$ and $n \mapsto f(a_2(n))$ converge*
If we can propose two convergence seqence a_1 and a_2 which codomain is in the domain of a function f exculding its c, where $\lim a_1 = \lim a_2 = c$, if

$$\lim(n \mapsto f(a_1(n))) \neq \lim(n \mapsto f(a_2(n))) \implies \lim_c(f) \text{ does not exist}$$

Proof. **TODO** □

Corollary 3.2 (Linearity of Function Limits). [cor:function-limit-linearity] *Let (X, d_X) be a metric space, let $A \subseteq X$, and let $f, g : A \to \mathbb{R}$, where $\mathbb{R}$ has its usual distance. Suppose $c \in \text{LimDom}(f) \cap \text{LimDom}(g)$, with $\lim_c(f) = L$ and $\lim_c(g) = M$. For $\alpha, \beta \in \mathbb{R}$, the following limit exists and*

$$\lim_c(x \mapsto \alpha \cdot f(x) + \beta \cdot g(x)) = \alpha \cdot L + \beta \cdot M$$

Proof. **TODO** □

Corollary 3.3 (Product of Function Limits). [cor:function-limit-product] *Let (X, d_X) be a metric space, let $A \subseteq X$, and let $f, g : A \to \mathbb{R}$, where $\mathbb{R}$ has its usual distance. Suppose $c \in \text{LimDom}(f) \cap \text{LimDom}(g)$, with $\lim_c(f) = L$ and $\lim_c(g) = M$. The following limit exists and*

$$\lim_c(x \mapsto f(x) \cdot g(x)) = L \cdot M$$

Proof. **TODO** □

Corollary 3.4 (Quotient of Function Limits). [cor:function-limit-quotient] *Let (X, d_X) be a metric space, let $A \subseteq X$, and let $f, g : A \to \mathbb{R}$, where $\mathbb{R}$ has its usual distance. Suppose $c \in \text{LimDom}(f) \cap \text{LimDom}(g)$, with $\lim_c(f) = L$ and $\lim_c(g) = M$. If $M \neq 0$ and $\forall x \in A,\ g(x) \neq 0$, the following limit exists and*

$$\lim_c\left(x \mapsto \frac{f(x)}{g(x)}\right) = \frac{L}{M}$$

Proof. **TODO** □

4 One-Sided Limits

Definition 4.1 (Right-Hand Limit)**.** [def:right-hand-limit] Let $A \subseteq \mathbb{R}$, let (B, d_B) be a metric space, and let $f : A \to B$. Let $c \in \mathbb{R}$ satisfy

$$\forall r \in \mathbb{R}_+, \quad \exists x \in A, \quad 0 < x - c < r$$

We say that f has *right-hand limit* $\ell \in B$ at c if

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall x \in A \cap (c, c+\delta), \, d_B(f(x), \ell) < \varepsilon$$

Definition 4.2 (Left-Hand Limit)**.** [def:left-hand-limit] Let $A \subseteq \mathbb{R}$, let (B, d_B) be a metric space, and let $f : A \to B$. Let $c \in \mathbb{R}$ satisfy

$$\forall r \in \mathbb{R}_+, \quad \exists x \in A, \quad 0 < c - x < r$$

We say that f has *left-hand limit* $\ell \in B$ at c if

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_-, \forall x \in A \cap (c+\delta, c), \, d_B(f(x), \ell) < \varepsilon$$

Theorem 4.1 (Uniqueness of One-Sided Limits)**.** [thm:one-sided-limit-uniqueness] *Under the assumptions of the right-hand or left-hand definition, the corresponding one-sided limit, if it exists, is unique*

Proof. The argument is the same as for two-sided limits, choosing x in the shared one-sided interval using the corresponding accumulation assumption □

Definition 4.3 (Right-Hand Limit Domain)**.** [def:right-hand-limit-domain] Let $A \subseteq \mathbb{R}$ and let (B, d_B) be a metric space

$$\mathrm{LimDom}_+ : (A \to B) \to \mathcal{P}(\mathbb{R})$$

$$\mathrm{LimDom}_+(f) \equiv \left\{ c \in \mathbb{R} \;\middle|\; \begin{array}{c} \forall r \in \mathbb{R}_+, \quad A \cap (c, c+r) \neq \emptyset \\ \wedge \\ \exists \ell \in B, \quad \forall \varepsilon \in \mathbb{R}_+, \quad \exists \delta \in \mathbb{R}_+, \\ \forall x \in A \cap (c, c+\delta), \quad d_B(f(x), \ell) < \varepsilon \end{array} \right\}$$

`LaTeX: \LimDomRight (f)`

Definition 4.4 (Left-Hand Limit Domain)**.** [def:left-hand-limit-domain] Let $A \subseteq \mathbb{R}$ and let (B, d_B) be a metric space

$$\mathrm{LimDom}_- : (A \to B) \to \mathcal{P}(\mathbb{R})$$

$$\mathrm{LimDom}_-(f) \equiv \left\{ c \in \mathbb{R} \;\middle|\; \begin{array}{c} \forall r \in \mathbb{R}_+, \quad A \cap (c-r, c) \neq \emptyset \\ \wedge \\ \exists \ell \in B, \quad \forall \varepsilon \in \mathbb{R}_+, \quad \exists \delta \in \mathbb{R}_-, \\ \forall x \in A \cap (c+\delta, c), \quad d_B(f(x), \ell) < \varepsilon \end{array} \right\}$$

`LaTeX: \LimDomLeft (f)`

Proposition 4.1 (Well-Definedness of One-Sided Limits)**.** [prop:one-sided-limit-well-defined] *For every $f : A \to B$ and $c \in \mathrm{LimDom}_+(f)$, there is a unique right-hand limit in B. The same holds for $c \in \mathrm{LimDom}_-(f)$ and the left-hand limit*

Proof. Existence follows from the definitions of the right-hand and left-hand limit domains. Uniqueness follows from the one-sided uniqueness theorem □

Definition 4.5 (Right-Hand Limit Operator). [def:right-hand-limit-op] The operator

$$\lim_{+} : \Pi f : (A \to B), \quad \mathrm{LimDom}_{+}(f) \to B$$

sends f and $c \in \mathrm{LimDom}_{+}(f)$ to the unique right-hand limit of f at c

`LaTeX: \rightfnlim (f)(c)`

Definition 4.6 (Left-Hand Limit Operator). [def:left-hand-limit-op] The operator

$$\lim_{-} : \Pi f : (A \to B), \quad \mathrm{LimDom}_{-}(f) \to B$$

sends f and $c \in \mathrm{LimDom}_{-}(f)$ to the unique left-hand limit of f at c

`LaTeX: \leftfnlim (f)(c)`

Convention 4.1 (One-Sided Limit Notation). [conv:one-sided-limit-notation] For $c \in \mathrm{LimDom}_{+}(f)$ or $c \in \mathrm{LimDom}_{-}(f)$, respectively, write

$$\lim_{c^+}(f) \equiv \lim_{+}(f)(c), \qquad \lim_{c^-}(f) \equiv \lim_{-}(f)(c)$$

`LaTeX: \poslim {3}(f), \neglim {3}(f)`

Theorem 4.2 (Two-Sided and One-Sided Limits). [thm:two-sided-one-sided-limits] *Let $A \subseteq \mathbb{R}$, let (B, d_B) be a metric space, and let $f : A \to B$. Suppose $c \in \mathbb{R}$ is a limit point of both $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then f has a two-sided limit at c if and only if its right-hand and left-hand limits both exist and are equal. In that case,*

$$\lim_{c}(f) = \lim_{c^+}(f) = \lim_{c^-}(f)$$

Proof. **TODO:** Prove both directions □

Theorem 4.3 (Unequal One-Sided Limits). [thm:unequal-one-sided-limits] *Let $A \subseteq \mathbb{R}$, let (B, d_B) be a metric space, and let $f : A \to B$. If $c \in \mathrm{LimDom}_{+}(f) \cap \mathrm{LimDom}_{-}(f)$ and*

$$\lim_{c^+}(f) \neq \lim_{c^-}(f)$$

then f has no two-sided limit at c, that is, $c \notin \mathrm{LimDom}(f)$

Proof. **TODO:** Use the equivalence between two-sided and equal one-sided limits □

Remark 4.1 (Error Bounds for One-Sided Limits). [rem:one-sided-error-bounds] **TODO:** Adapt the shared error-bound method to the right-hand and left-hand input regions

5 Limits at Infinity

Definition 5.1 (Limit at Positive Infinity). [def:limit-at-positive-infinity] Let $A \subseteq \mathbb{R}$, let (B, d_B) be a metric space, and let $f : A \to B$. Assume A is unbounded above, that is,

$$\forall t \in \mathbb{R}, \quad \exists x \in A, \quad x > t$$

We say that f has *limit* $\ell \in B$ *at* $+\infty$ if

$$\forall \varepsilon \in \mathbb{R}_{+}, \exists n \in \mathbb{R}, \forall x \in A, \; x > n \implies d_B(f(x), \ell) < \varepsilon$$

Definition 5.2 (Limit at Negative Infinity). [def:limit-at-negative-infinity] Let $A \subseteq \mathbb{R}$, let (B, d_B) be a metric space, and let $f : A \to B$. Assume A is unbounded below, that is,

$$\forall t \in \mathbb{R}, \quad \exists x \in A, \quad x < t$$

We say that f has *limit* $\ell \in B$ *at* $-\infty$ if

$$\forall \varepsilon \in \mathbb{R}_+, \exists n \in \mathbb{R}, \forall x \in A, \, x < n \implies d_B(f(x), \ell) < \varepsilon$$

Theorem 5.1 (Uniqueness of Limits at Infinity). [thm:limit-at-infinity-uniqueness] *Under the assumptions of the definition at positive infinity or negative infinity, the corresponding limit, if it exists, is unique*

Proof. Suppose $\ell_1, \ell_2 \in B$ are limits in the same direction. Fix $\varepsilon > 0$ and choose the corresponding thresholds n_1, n_2. At $+\infty$, unboundedness above supplies $x \in A$ with $x > \max(n_1, n_2)$; at $-\infty$, unboundedness below supplies $x \in A$ with $x < \min(n_1, n_2)$. In either case, the triangle inequality gives

$$d_B(\ell_1, \ell_2) \leq d_B(\ell_1, f(x)) + d_B(f(x), \ell_2) < 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, the distance is zero, so $\ell_1 = \ell_2$ by metric separation □

Definition 5.3 (Positive Infinity Limit Domain). [def:positive-infinity-limit-domain] Let $A \subseteq \mathbb{R}$ and let (B, d_B) be a metric space. Define $\mathrm{Conv}_{+\infty}(A, B) \in \mathcal{P}(A \to B)$ by

$$\mathrm{Conv}_{+\infty}(A, B) \equiv \left\{ f : A \to B \;\middle|\; \begin{array}{c} \forall t \in \mathbb{R}, \quad \exists x \in A, \quad x > t \\ \wedge \\ \exists \ell \in B, \quad \forall \varepsilon \in \mathbb{R}_+, \quad \exists n \in \mathbb{R}, \\ \forall x \in A, \quad x > n \implies d_B(f(x), \ell) < \varepsilon \end{array} \right\}$$

`LaTeX: \ConvPosInf (A,B)`

Definition 5.4 (Negative Infinity Limit Domain). [def:negative-infinity-limit-domain] Let $A \subseteq \mathbb{R}$ and let (B, d_B) be a metric space. Define $\mathrm{Conv}_{-\infty}(A, B) \in \mathcal{P}(A \to B)$ by

$$\mathrm{Conv}_{-\infty}(A, B) \equiv \left\{ f : A \to B \;\middle|\; \begin{array}{c} \forall t \in \mathbb{R}, \quad \exists x \in A, \quad x < t \\ \wedge \\ \exists \ell \in B, \quad \forall \varepsilon \in \mathbb{R}_+, \quad \exists n \in \mathbb{R}, \\ \forall x \in A, \quad x < n \implies d_B(f(x), \ell) < \varepsilon \end{array} \right\}$$

`LaTeX: \ConvNegInf (A,B)`

Proposition 5.1 (Well-Definedness of Limits at Infinity). [prop:limit-at-infinity-well-defined] *A function $f : A \to B$ belongs to $\mathrm{Conv}_{+\infty}(A, B)$ if and only if it has a unique limit at $+\infty$. Likewise, $f \in \mathrm{Conv}_{-\infty}(A, B)$ if and only if it has a unique limit at $-\infty$*

Proof. Membership is equivalent to existence by the definitions of the positive infinity and negative infinity limit domains. Uniqueness follows from the uniqueness theorem □

Convention 5.1 (Limit Notation at Infinity). [conv:limit-at-infinity-notation] **TODO:** Introduce the notation for limits as the input tends to positive or negative infinity

Remark 5.1 (Error Bounds for Limits at Infinity). [rem:infinity-error-bounds] **TODO:** Adapt the shared error-bound method using a real input threshold M(t)

6 A Systematic Method for Proving Limits

Remark 6.1 (Board Comparison). [rem:systematic-limit-comparison]

	Sequence	**Infinity $+\infty$**	**One-Sided c^+**	**Function**
Space	(B, d_B)	$\substack{(\mathbb{R},d)\\(B,d_B)}$	$\substack{(\mathbb{R},d)\\(B,d_B)}$	$\substack{(X,d_X)\\(B,d_B)}$
Map	$a : \mathbb{N}_1 \to B$	$\substack{f:A\to B\\A\subseteq\mathbb{R}}$	$\substack{f:A\to B\\A\subseteq\mathbb{R}}$	$\substack{f:A\to B\\A\subseteq X}$
Arbitrary tolerance	$\forall \varepsilon \in \mathbb{R}_+$	$\forall \varepsilon \in \mathbb{R}_+$	$\forall \varepsilon \in \mathbb{R}_+$	$\forall \varepsilon \in \mathbb{R}_+$
Restrictor	$\exists N \in \mathbb{N}_1$	$\exists n \in \mathbb{R}$	$\exists \delta \in \mathbb{R}_+$	$\exists \delta \in \mathbb{R}_+$
Input	$\forall k \in \mathbb{N}_1$	$\forall x \in A$	$\forall x \in A$	$\forall x \in A$
Input restriction	$k \geq N$	$x > n$	$c < x < c + \delta$	$0 < d_X(x, c) < \delta$
Output requirement	$d_B(a(k), \ell) < \varepsilon$	$d_B(f(x), \ell) < \varepsilon$	$d_B(f(x), \ell) < \varepsilon$	$d_B(f(x), \ell) < \varepsilon$

Convention 6.1 (Error Bounds for Sequences)**.** [conv:error-bound-method] Let $a : \mathbb{N}_1 \to B$, where (B, d_B) is a metric space, and let $\ell \in B$.

Step 1. Compute the *error*, the distance from a term to the proposed limit:

$$E(n) \equiv d_B(a(n), \ell)$$

Step 2. Find an estimate valid for every $n \in \mathbb{N}_1$:

$$n > s \implies E(n) \leq H(s)$$

Note any restrictions on s

Step 3. Choose $s(t) > 0$ satisfying those restrictions for every $t > 0$, and set

$$\rho(t) \equiv \lfloor s(t) \rfloor + 1, \qquad g(t) \equiv H(s(t))$$

Then $\rho : \mathbb{R}_+ \to \mathbb{N}_1$, and

$$n \geq \rho(t) \implies n > s(t) \implies E(n) \leq g(t)$$

Step 4. Check that g is a challenge function. Given $\varepsilon > 0$, choose $t > 0$ with $g(t) < \varepsilon$; then $n \geq \rho(t)$ gives $E(n) \leq g(t) < \varepsilon$, proving convergence to ℓ

Example 6.1 (Sequence)**.** [eg:error-bound-sequence] Let

$$a : \mathbb{N}_1 \to \mathbb{R} \equiv k \mapsto \frac{1}{k}$$

Show that $\lim(a) = 0$

Proof. **Step 1.** The proposed limit is 0, so for $n \in \mathbb{N}_1$,

$$E(n) \equiv d(a(n), 0) = \left| \frac{1}{n} - 0 \right| = \frac{1}{n}$$

Step 2. For $n > s > 0$,

$$E(n) = \frac{1}{n} < \frac{1}{s}$$

Thus take $H(s) \equiv 1/s$, with restriction $s > 0$

Step 3. Choose $s(t) \equiv t$, and define

$$\rho : \mathbb{R}_+ \to \mathbb{N}_1 \equiv t \mapsto \lfloor t \rfloor + 1, \qquad g : \mathbb{R}_+ \to \mathbb{R}_+ \equiv t \mapsto \frac{1}{t}$$

Then $n \geq \rho(t)$ implies $n > t$, so

$$\forall t \in \mathbb{R}_+, \forall n \in \mathbb{N}_1, \quad n \geq \rho(t) \implies E(n) < g(t)$$

Step 4. By the reciprocal lemma, g is a challenge function, so

$$\lim(a) = 0$$

□

Example 6.2 (Sequence with a Threshold Function)**.** [eg:error-bound-sequence-helper] Let

$$a : \mathbb{N}_1 \to \mathbb{R} \equiv k \mapsto \frac{1}{k^2 - 2}$$

Show that $\lim(a) = 0$ by choosing a challenge function g and a threshold function $\rho : \mathbb{R}_+ \to \mathbb{N}_1$

Proof. **Step 1.** The proposed limit is 0, so for $n \in \mathbb{N}_1$,

$$E(n) \equiv d(a(n), 0) = \left| \frac{1}{n^2 - 2} - 0 \right| = \left| \frac{1}{n^2 - 2} \right|$$

Step 2. **TODO**

Step 3. **TODO**

Step 4. **TODO**

□

Convention 6.2 (Error Bounds at Infinity)**.** [conv:error-bound-infinity] Use the setup for a limit at positive infinity. Find a challenge function g bounding the output error on a tail. A direct cutoff is $x > t$, for $t > 0$. If a separate threshold is useful, choose $\rho : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\forall t \in \mathbb{R}_+, \forall x \in A, \quad x > \rho(t) \implies d_B(f(x), \ell) \leq g(t)$$

At negative infinity, use $x < \rho(t)$ instead, with direct cutoff $x < -t$

Example 6.3 (Limit at Positive Infinity)**.** [eg:error-bound-positive-infinity] Let

$$f : \mathbb{R}_+ \to \mathbb{R} \equiv x \mapsto \frac{1}{x}$$

Show that f has limit 0 at $+\infty$

Proof. **TODO** □

Example 6.4 (Limit at Infinity with a Threshold Function)**.** [eg:error-bound-infinity-helper] Let

$$f : (1, \infty) \to \mathbb{R} \equiv x \mapsto \frac{1}{x - 1}$$

Show that f has limit 0 at $+\infty$ by choosing a challenge function g and a threshold function $\rho : \mathbb{R}_+ \to \mathbb{R}$

Proof. **TODO** □

Convention 6.3 (Error Bounds for One-Sided Limits). [conv:error-bound-one-sided] Use the right-hand limit setup. Find a challenge function g bounding the output error on a right-hand interval. A direct radius is $t > 0$. If a separate radius is useful, choose $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall t \in \mathbb{R}_+, \forall x \in A \cap (c, c + \rho(t)), \quad d_B(f(x), \ell) \leq g(t)$$

For a left-hand limit, use $\rho : \mathbb{R}_+ \to \mathbb{R}_-$ and the interval $(c + \rho(t), c)$, with direct offset $-t$

Example 6.5 (Right-Hand Limit). [eg:error-bound-right-hand-limit] Let

$$f : \mathbb{R}_{\geq 0} \to \mathbb{R} \equiv x \mapsto x^2$$

Show that $\lim_{0^+}(f) = 0$

Proof. **TODO** □

Example 6.6 (Right-Hand Limit with a Radius Function). [eg:error-bound-right-hand-helper] Let

$$f : (0, 1) \to \mathbb{R} \equiv x \mapsto \frac{x}{1 - x}$$

Show that $\lim_{0^+}(f) = 0$ by choosing a challenge function g and a radius function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$

Proof. **TODO** □

Convention 6.4 (Error Bounds for Two-Sided Limits). [conv:error-bound-two-sided] Use the two-sided limit setup. Find a challenge function g bounding the output error on a punctured ball. A direct radius is $t > 0$. If a separate radius is useful, choose $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall t \in \mathbb{R}_+, \forall x \in A, \quad 0 < d_X(x, c) < \rho(t) \implies d_B(f(x), \ell) \leq g(t)$$

Example 6.7 (Two-Sided Limit with a Direct Radius). [eg:error-bound-two-sided-direct] Let

$$f : \mathbb{R} \to \mathbb{R} \equiv x \mapsto 3 \cdot x + 1$$

Show that $\lim_2(f) = 7$

Proof. **TODO** □

Example 6.8 (Two-Sided Function Limit). [eg:error-bound-two-sided-limit] Let

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \equiv x \mapsto \frac{x - 1}{x}$$

Show that $\lim_1(f) = 0$ by choosing a challenge function g and a radius function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$

Proof. **TODO** □

Remark 6.2 (Local Input Restrictions). [rem:ball-radius-parameter] **TODO:** Record any local input restriction needed in the preceding example

Remark 6.3 (Comparison of the Error Bounds). [rem:error-bound-method-comparison]

	Sequence Infinity $+\infty$		**One-Sided c^+**	**Function**
Direct restrictor	**TODO**	**TODO**	**TODO**	**TODO**
Direct bound $g(t)$	**TODO**	**TODO**	**TODO**	**TODO**
Helper $\rho(t)$	**TODO**	**TODO**	**TODO**	**TODO**
Input with $\rho(t)$	**TODO**	**TODO**	**TODO**	**TODO**
Helper bound $g(t)$	**TODO**	**TODO**	**TODO**	**TODO**