Lecture 5: Series

Mathematical Analysis

1 Subsequence

Definition 1.1 (Subsequence). [def:subsequence] If a sequence $n : \mathbb{N}_1 \to \mathbb{N}_1$ is strictly increasing, then the sequence $b : \mathbb{N}_1 \to F \equiv k \mapsto a(n(k))$ is called a *subsequence* of $a : \mathbb{N}_1 \to F$

Theorem 1.1 (Subsequence Index Grows). [thm:subsequence-index-grows] *If $b : \mathbb{N}_1 \to F \equiv k \mapsto a(n(k))$ is a subsequence of $a : \mathbb{N}_1 \to F$, then $\forall k \in \mathbb{N}_1, n(k) \geq k$*

Proof. Since $n : \mathbb{N}_1 \to \mathbb{N}_1$ is strictly increasing, we have $n(1) < n(2) < n(3) < \cdots$
Notice that $n(1) \geq 1$, since $n(1) \in \mathbb{N}_1$
Also, if $n(k) \geq k$, then $n(k+1) > n(k) \geq k$, so $n(k+1) \geq k+1$
By induction, we have $\forall k \in \mathbb{N}_1, n(k) \geq k$ □

Theorem 1.2 (Subsequence of a Subsequence). [thm:subsequence-of-subsequence] *A subsequence of a subsequence is still a subsequence of the original sequence*

Remark 1.1 (Intuition: Picking Forward). [rem:subsequence-intuition] A subsequence can be understand as picking element from a sequence, however during picking we can only go forward, cannot go back to pick an element or pick an element twice

Theorem 1.3 (Limit of a Convergent Subsequence). [thm:convergent-subsequence-limit] *The limit of a subsequence is the same as the limit of the original sequence*

Proof. **TODO** □

Corollary 1.1 (Diverges via Two Limits). [cor:divergence-via-subsequences] *If a sequence have two subsequences with different limits, then the sequence diverges*

Proof. **TODO** □

Theorem 1.4 (Sequential Compactness). [thm:sequential-compactness] *Let (X, d) be a metric space. A subset $K \subseteq X$ is compact in the induced topology if and only if every sequence $a : \mathbb{N}_1 \to K$ has a subsequence $b : \mathbb{N}_1 \to K$ and a point $\ell \in K$ such that*

$$\forall \varepsilon \in \mathbb{R}_+, \exists N \in \mathbb{N}_1, \forall k \in \mathbb{N}_1, \quad k \geq N \implies d(b_k, \ell) < \varepsilon$$

Proof. **TODO:** Proof □

2 Bolzano-Weierstrass Theorem

Theorem 2.1 (Bolzano-Weierstrass Theorem). [thm:bw] *Every bounded sequence $a : \mathbb{N}_1 \to F$ where F is a complete ordered field has a convergent subsequence*

Proof. **TODO** □

3 Cauchy Criterion

Definition 3.1 (Cauchy Sequence). [def:cauchy-sequence] Let F be an ordered field. $a : \mathbb{N}_1 \to F$ is a *Cauchy sequence* if

$$\forall \varepsilon \in F_+, \exists N \in \mathbb{N}_1, \forall n, m \in \mathbb{N}_1 \, n > N \text{ and } m > N \implies d(a_n, a_m) < \varepsilon$$

Theorem 3.1 (Cauchy Criterion). [thm:cauchy-criterion] *If F is complete ordered field, then $a : \mathbb{N}_1 \to F$ is a Cauchy sequence if and only if it converges*

Proof. **TODO** □

Theorem 3.2 (Cauchy Implies Bounded). [thm:cauchy-bounded] *Cauchy sequence are bounded*

Proof. **TODO** □

4 Series

Definition 4.1 (Series Operator). [def:series-operator] Let F be a field. The *series operator*

$$\text{series} : (\mathbb{N}_1 \to F) \to (\mathbb{N}_1 \to F)$$

is defined recursively by

$$\text{series}(a)(1) \equiv a(1), \qquad \text{series}(a)(n+1) \equiv \text{series}(a)(n) + a(n+1)$$

Definition 4.2 (Series). [def:series] Let F be a field and $a : \mathbb{N}_1 \to F$. The *series of a* is the sequence

$$\text{series}(a) : \mathbb{N}_1 \to F,$$

obtained by applying the series operator to a. Its n-th term is the partial sum $\sum_1^n a$

Definition 4.3 (Convergent Series). [def:convergent-series] Let F be a field and $a : \mathbb{N}_1 \to F$. The series of a is *convergent* if $\text{series}(a)$ is a convergent sequence

Definition 4.4 (Infinite Summation). [def:infinite-series] Let F be a field and $a : \mathbb{N}_1 \to F$ with a convergent series. The *infinite summation* of a is

$$\sum a \equiv \lim(\text{series}(a))$$

`LaTeX: \infsum (a)`

Theorem 4.1 (Linearity of Infinite Summation). [thm:linearity-of-infinite-summation] *Let F be a field and $a, b : \mathbb{N}_1 \to F$ with convergent series. Then*

$$\sum(a+b) = \sum a + \sum b, \qquad \sum(\lambda a) = \lambda \sum a$$

Theorem 4.2. *A series of a sequence $a : \mathbb{N}_1 \to F$ where F is a complete ordered field converges only if a converges to 0*

Proof. **TODO**(Use cauchy criterion to prove this theorem) □

5 Comparison Test for Sequence

Lemma 5.1 (Series with Non-Negative Terms is Increasing). [lem:series-nonneg-increasing] *Let F be an ordered field and $a : \mathbb{N}_1 \to F$ with $a(n) \geq 0$ for all $n \in \mathbb{N}_1$. Then* series(a) *is an increasing sequence*

Lemma 5.2 (Series with Positive Terms is Strictly Increasing). [lem:series-positive-strictly-increasing] *Let F be an ordered field and $a : \mathbb{N}_1 \to F$ with $a(n) > 0$ for all $n \in \mathbb{N}_1$. Then* series(a) *is a strictly increasing sequence*

Corollary 5.1 (Comparison Test for Series). [cor:comparison-test] *Let F be a complete ordered field, $a, b : \mathbb{N}_1 \to F$ with $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}_1$. If the series of b is convergent, then the series of a is convergent and $\sum a \leq \sum b$. Follows by applying theorem monotone comparison increasing to the partial sum sequences* series(a) *and* series(b)*, both of which are increasing by lemma 5.1*

Theorem 5.1 (Absolute Convergence Implies Convergence). [thm:absolute-convergence] *Let F be a complete ordered field and $a : \mathbb{N}_1 \to F$. If the series of $|a|$ (where $|a|(n) \equiv |a(n)|$) is convergent, then the series of a is convergent*

Proof. Since $0 \leq a(n) + |a(n)| \leq 2|a(n)|$, the comparison test gives convergence of the series of $n \mapsto a(n) + |a(n)|$, and subtracting the convergent series of $n \mapsto |a(n)|$ gives convergence of the series of a by linearity □

Theorem 5.2 (Convergence of Nonnegative Series). [thm:nonnegative-series-criterion] *Let F be a complete ordered field, and let $a : \mathbb{N}_1 \to F$ satisfy $a(n) \geq 0$ for every $n \in \mathbb{N}_1$.*
The series of a is convergent if and only if P is bounded above in F. In that case,

$$\sum(a) = \sup(P), \quad P \equiv \left\{ \sum_{1}^{n}(a) : n \in \mathbb{N}_1 \right\}$$

Proof. Suppose P is bounded above and series(a) converges. Write $M = \sup(P)$ and $L = \lim(\text{series}(a))$. According to the ε-criterion for the supremum,

$$\forall \varepsilon_1 \in F_+, \exists s \in P, \quad |M - s| < \varepsilon_1$$

According to the definition of a limit,

$$\forall \varepsilon_2 \in F_+, \exists N \in \mathbb{N}_1, \forall n \in \mathbb{N}_1, \quad n \geq N \implies |\text{series}(a)(n) - L| < \varepsilon_2$$

Fix arbitrary $\varepsilon_1, \varepsilon_2 \in F_+$ and corresponding witnesses s, N, and let $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$.

$$\forall s \in P, \exists k : \mathbb{N}_1, s = \text{series}(a)(k)$$

Set $n = \max(k, N)$. Since the partial sums are increasing and M is an upper bound,

$$s = \text{series}(a)(k) \leq \text{series}(a)(n) \leq M$$

Hence

$$|M - \text{series}(a)(n)| = M - \text{series}(a)(n) \leq M - s < \varepsilon_1 \leq \varepsilon$$

Also $n \geq N$, so the convergence bound applies at this same index
Thus

$$|M - \text{series}(a)(n)| < \varepsilon, \qquad |\text{series}(a)(n) - L| < \varepsilon$$

By the triangle inequality,

$$|M - L| \leq |M - \text{series}(a)(n)| + |\text{series}(a)(n) - L| < 2 \cdot \varepsilon$$

The function $g(\varepsilon) \equiv 2 \cdot \varepsilon$ is a linear challenge function. Taking $\varepsilon_1 = \varepsilon_2 = \varepsilon$ shows that $|M - L| < g(\varepsilon)$ for every $\varepsilon \in F_+$. By the equality criterion with a challenge function,

$$\sup(P) = \lim(\text{series}(a)) = \sum(a)$$

□

Remark 5.1 (Two Methods for Checking Series Convergence). [rem:two-methods-series] Depending on the sign structure of the terms of a:

- *Single-sign* (all $a(n) \geq 0$ or all $a(n) \leq 0$): apply the Comparison Test for Series directly. Monotonicity of series(a) comes from lemma 5.1 (or the sign-flipped version)
- *Mixed-sign*: apply Absolute Convergence first, reducing the question to series($|a|$) (nonnegative), where the Comparison Test applies. For alternating terms, the alternating-series test may apply even when the series of absolute values diverges

6 Grouping of Series

Definition 6.1 (Grouping Index). [def:grouping-index] A *grouping index* is a sequence $i : \mathbb{N}_1 \to \mathbb{N}_1$ satisfying

$$i(1) = 1, \qquad \forall n \in \mathbb{N}_1, \quad i(n) < i(n+1)$$

Definition 6.2 (Grouping of Series). [def:series-grouping] Let F be a field and i a grouping index. Define

$$\text{group}(i) : (\mathbb{N}_1 \to F) \to (\mathbb{N}_1 \to F) \equiv a \mapsto n \mapsto \sum_{i(n)}^{i(n+1)-1} (a)$$

The grouped series of a is the series of $\text{group}(i)(a)$

Lemma 6.1 (Partial Sums of a Grouped Series). [lem:grouped-partial-sums] *Let F be a field, i a grouping index, and $a : \mathbb{N}_1 \to F$. Then*

$$\forall m \in \mathbb{N}_1, \quad \sum_{1}^{m} \big(\text{group}(i)(a)\big) = \sum_{1}^{i(m+1)-1} (a)$$

Proof. By the definition of grouping and repeated application of splitting of summation, the first m blocks combine into the sum from $i(1) = 1$ to $i(m+1) - 1$ □

Theorem 6.1 (Grouping Preserves Convergence). [thm:grouping-preserves-convergence]

Theorem 6.2 (Converse for Bounded Block Lengths). [thm:grouping-converse-bounded-blocks]

Corollary 6.1 (Grouping of Nonnegative Series). [cor:nonnegative-series-grouping]

Example 6.1 (Converse Fails for Unbounded Blocks). [eg:grouping-converse-unbounded-blocks]

7 Special Series

Definition 7.1 (Harmonic Series). [def:harmonic-series] The *harmonic series* is the series of the sequence

$$a : \mathbb{N}_1 \to \mathbb{R} \equiv n \mapsto \frac{1}{n}$$

Theorem 7.1 (Harmonic Series Diverges). [thm:harmonic-series-diverges] *The harmonic series is divergent*

Proof. **TODO** □

Definition 7.2 (Reciprocal Squares Series). [def:reciprocal-squares-series] The *reciprocal squares series* is the series of the sequence

$$a : \mathbb{N}_1 \to \mathbb{R} \equiv n \mapsto \frac{1}{n^2}$$

Theorem 7.2 (Reciprocal Squares Series Converges). [thm:reciprocal-squares-converges] *The reciprocal squares series is convergent*

Proof. **TODO** □

Definition 7.3 (Alternating Series). [def:alternating-series] Let $b : \mathbb{N}_1 \to \mathbb{R}_{\geq 0}$ and $\sigma \in \{-1, 1\}$. The series of the sequence

$$a : \mathbb{N}_1 \to \mathbb{R} \equiv n \mapsto \sigma \cdot (-1)^{n+1} \cdot b(n)$$

is called an *alternating series*

Theorem 7.3 (Alternating-Series Test). [thm:alternating-series-test] *Let* $b : \mathrm{Conv}(\mathbb{R})$ *and* $\sigma \in \{-1, 1\}$. *If*

$$\forall n \in \mathbb{N}_1, \quad 0 \leq b(n+1) \leq b(n), \qquad \lim(b) = 0$$

then the alternating series of $n \mapsto \sigma \cdot (-1)^{n+1} \cdot b(n)$ *is convergent*

Proof. **TODO** □

Example 7.1 (Alternating Harmonic Series). [eg:alternating-harmonic-series] The *alternating harmonic series* is the series of the sequence

$$a : \mathbb{N}_1 \to \mathbb{R} \equiv n \mapsto \frac{(-1)^{n+1}}{n}$$

It is convergent, while the series of $n \mapsto |a(n)|$ is the divergent harmonic series

Proof. **TODO** □