# Lecture 4: Limit of Sequence

Mathematical Analysis

## 1 Convergent Sequences

**Definition 1.1** (Convergent Sequence)**.** [def:convergent-sequence] A sequence $a : \mathbb{N}_1 \to F$, where $F$ is a ordered field, is said to be *convergent* if

$$\exists \ell \in F,\, \forall \varepsilon \in F_+,\, \exists N \in \mathbb{N}_1,\, \forall n \in \mathbb{N}_1,\, n \geq N,\, d(a_n, \ell) < \varepsilon$$

*Remark* 1.1 (The $\varepsilon$-$N$ Challenge). [rem:epsilon-n-challenge] A value $\ell$ passes the $\varepsilon$-$N$ challenge if for any proposed $\varepsilon > 0$, there exists an index $N \in \mathbb{N}_1$ such that for all $n \in \mathbb{N}_1$ with $n \geq N$, $d(a_n, \ell) < \varepsilon$

*Remark* 1.2. In the The $\varepsilon$-$N$ Challenge, the value of $N$ may depend on the choice of $\varepsilon$. And the value $N$ may be not unique (ie, $N' > N$ also works)

**Theorem 1.1** (Uniqueness of Limit)**.** [thm:limit-uniqueness] *Let $F$ be an ordered field, $a : \mathbb{N}_1 \to F$, and $\ell_1, \ell_2 \in F$. Then*

$$\begin{gathered}\big(\forall \varepsilon \in F_+,\, \exists N \in \mathbb{N}_1,\, \forall n \in \mathbb{N}_1,\, n \geq N,\, d(a_n, \ell_1) < \varepsilon\big) \\ \wedge \\ \big(\forall \varepsilon \in F_+,\, \exists N \in \mathbb{N}_1,\, \forall n \in \mathbb{N}_1,\, n \geq N,\, d(a_n, \ell_2) < \varepsilon\big) \\ \implies \ell_1 = \ell_2\end{gathered}$$

*Proof.* Assume for contradiction that $\ell_1 \neq \ell_2$. Set

$$\varepsilon = \frac{d(\ell_1, \ell_2)}{2}.$$

Since $\ell_1 \neq \ell_2$, we have $\varepsilon > 0$
By the hypothesis for $\ell_1$, there exists $N_1 \in \mathbb{N}_1$ such that for all $n \geq N_1$, $d(a_n, \ell_1) < \varepsilon$. Similarly, there exists $N_2 \in \mathbb{N}_1$ such that for all $n \geq N_2$, $d(a_n, \ell_2) < \varepsilon$
Let $N = \max(N_1, N_2)$. For any $n \geq N$, both hypotheses hold:

$$d(a_n, \ell_1) < \varepsilon \quad \text{and} \quad d(a_n, \ell_2) < \varepsilon$$

By the triangle inequality for distance applied via the intermediate point $a_n$,

$$d(\ell_1, \ell_2) \leq d(\ell_1, a_n) + d(a_n, \ell_2) = d(a_n, \ell_1) + d(a_n, \ell_2),$$

where the last equality uses symmetry of distance. Combining,

$$d(\ell_1, \ell_2) \leq d(a_n, \ell_1) + d(a_n, \ell_2) < 2\varepsilon = d(\ell_1, \ell_2),$$

a contradiction. Hence $\ell_1 = \ell_2$ □

*Remark* 1.3 (Compact Form Using $\exists!$). [rem:limit-uniqueness-compact] Combined with the existence from the definition of convergent sequence, this theorem states: if $a$ is convergent, then there exists a unique such $\ell$:

$$\exists! \ell \in F,\, \forall \varepsilon \in F_+,\, \exists N \in \mathbb{N}_1,\, \forall n \in \mathbb{N}_1,\, n \geq N,\, d(a_n, \ell) < \varepsilon.$$

Here $\exists!$ is shorthand for "existence and uniqueness"; it should be formally introduced in `lec01` as a FOL convention before being used

# 2 Convergence Challenge Functions

**Definition 2.1** (Convergence Challenge Function). [def:convergence-challenge-function] A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a *convergence challenge function* if it satisfies two conditions:

- *Positivity on positives*: $\forall \gamma \in \mathbb{R}_+,\ g(\gamma) > 0$
- *Arbitrarily small values*: $\forall \delta \in \mathbb{R}_+,\ \exists \gamma \in \mathbb{R}_+,\ g(\gamma) < \delta$

*Remark* 2.1 (Intuition: Covering Arbitrarily Small Positives). [rem:challenge-intuition] We only need arbitrarily small error bounds: a smaller bound automatically satisfies every larger tolerance

**Theorem 2.1** (Convergence Challenge Replacement). [thm:convergence-challenge-replacement] *Let $g$ be a convergence challenge function, and let $a : \mathbb{N}_1 \to X$ be a sequence in a metric space $(X, d)$. Then*

$$\big(\exists \ell \in X,\ \forall \varepsilon \in \mathbb{R}_+,\ \exists N \in \mathbb{N}_1,\ \forall n \in \mathbb{N}_1,\ n \geq N,\ d(a_n, \ell) < \varepsilon\big)$$

$$\iff$$

$$\big(\exists \ell \in X,\ \forall \varepsilon \in \mathbb{R}_+,\ \exists N \in \mathbb{N}_1,\ \forall n \in \mathbb{N}_1,\ n \geq N,\ d(a_n, \ell) < g(\varepsilon)\big)$$

*Proof.* Fix the witness $\ell \in X$ in either direction
($\Rightarrow$) For $\varepsilon > 0$, the positivity-on-positives condition gives $g(\varepsilon) > 0$. Apply the left-hand condition at the tolerance $g(\varepsilon)$
($\Leftarrow$) For $\varepsilon > 0$, the challenge-function property of $g$ gives $\gamma > 0$ with $g(\gamma) < \varepsilon$. The hypothesis at $\gamma$ gives $N$ such that

$$d(a_n, \ell) < g(\gamma) < \varepsilon \quad \text{for all } n \geq N$$

□

**Corollary 2.1** (Equality via Challenge Functions). [cor:equality-challenge-replacement] *Let $g$ be a convergence challenge function, and let $x, y \in X$ in a metric space $(X, d)$. Then*

$$x = y \iff \forall \varepsilon \in \mathbb{R}_+, \quad d(x, y) < g(\varepsilon)$$

*Proof.* ($\Rightarrow$) If $x = y$, then $d(x, y) = 0 < g(\varepsilon)$ for every $\varepsilon > 0$
($\Leftarrow$) If $x \neq y$, then $d(x, y) > 0$. The challenge-function property gives $\gamma > 0$ with $g(\gamma) < d(x, y)$, contradicting $d(x, y) < g(\gamma)$. Hence $x = y$ □

**Lemma 2.1** (Restricting to Small Tolerances). [lem:restricting-to-small-tolerances] *Fix $r \in \mathbb{R}_+$. The function*

$$g : \mathbb{R}_+ \to \mathbb{R}_+ \equiv \gamma \mapsto \min(\gamma, r/2)$$

*is a convergence challenge function*

*Proof.* For every $\gamma > 0$, $0 < g(\gamma) \leq r/2 < r$. For every $\delta > 0$, choose $\gamma \equiv \delta/2 > 0$; then $g(\gamma) \leq \delta/2 < \delta$ □

*Remark* 2.2 (Restricting the Tolerance). [rem:restricting-the-tolerance] Since $0 < g(\gamma) < r$ for $\gamma > 0$, challenge replacement lets us check $\forall\, 0 < \varepsilon < r$ instead of $\forall \varepsilon > 0$, for any fixed $r > 0$, eg, $\forall\, 0 < \varepsilon < 1$

**Lemma 2.2** (Linear Scaling). [lem:linear-is-challenge] *For any $c \in \mathbb{R}$ with $c > 0$, the function $g : \mathbb{R}_+ \to \mathbb{R}_+ \equiv \gamma \mapsto c \cdot \gamma$ is a convergence challenge function*

*Proof.* We verify both conditions
*Positivity on positives.* For any $\gamma \in \mathbb{R}$ with $\gamma > 0$, both $c > 0$ and $\gamma > 0$, so $g(\gamma) = c \cdot \gamma > 0$
*Arbitrarily small values.* Fix any $\delta \in \mathbb{R}$ with $\delta > 0$. We seek $\gamma \in \mathbb{R}$ with $\gamma > 0$ and $g(\gamma) < \delta$
Take $\gamma \equiv \delta/(c + 1)$, which is derived in remark 2.3. Since $\delta > 0$ and $c + 1 > 0$ (because $c > 0$), we have $\gamma > 0$
Now $g(\gamma) = c \cdot \gamma = c \cdot \delta/(c+1)$. Since $c < c+1$, we have $c/(c+1) < 1$, hence $c \cdot \delta/(c+1) < \delta$ □

*Remark* 2.3 (Deriving the Witness for Linear Scaling). [rem:linear-scaling-witness-derivation] To find the witness $\gamma$ used in the proof of lemma 2.2, we start from the target inequality $g(\gamma) < \delta$ and manipulate it to isolate $\gamma$:

$$g(\gamma) < \delta \iff c \cdot \gamma < \delta \iff \gamma < \frac{\delta}{c}.$$

Any $\gamma$ in the open interval $\left(0, \frac{\delta}{c}\right)$ will therefore satisfy $g(\gamma) < \delta$. To pick a concrete such $\gamma$, we use the fact that $c + 1 > c$, which gives

$$\frac{\delta}{c+1} < \frac{\delta}{c}.$$

Hence we may take $\gamma \equiv \delta/(c+1)$ as the witness

*Remark* 2.4 (Why $c \leq 0$ is Excluded). [rem:linear-scaling-cnonpos] If $c \leq 0$, then for any $\gamma \in \mathbb{R}$ with $\gamma > 0$, $g(\gamma) = c \cdot \gamma \leq 0$. This violates the positivity-on-positives condition, so $g$ is not a convergence challenge function

**Lemma 2.3** (Power). [lem:power-is-challenge] *For any positive integer $k$, the function $g : \mathbb{R}_+ \to \mathbb{R}_+ \equiv \gamma \mapsto \gamma^k$ is a convergence challenge function*

*Proof.* Both conditions follow by induction on $k$. Positivity: $\gamma^k > 0$ for $\gamma > 0$. For arbitrarily small values, take $\gamma \equiv \min(\delta/2,\, 1/2)$, so $\gamma > 0$ and $\gamma \leq 1$; induction gives $\gamma^k \leq \gamma \leq \delta/2 < \delta$ □

**Lemma 2.4** (Summation). [lem:sum-is-challenge] *Let $f, g$ be convergence challenge functions, and suppose both are monotone increasing (i.e. $x \leq y \implies f(x) \leq f(y)$, and similarly for $g$). Then $f + g$, defined by $(f + g)(\gamma) \equiv f(\gamma) + g(\gamma)$, is a convergence challenge function*

*Proof.* Positivity is immediate from $f(\gamma), g(\gamma) > 0$. For $\delta > 0$, pick $\gamma_1, \gamma_2 > 0$ with $f(\gamma_1), g(\gamma_2) < \delta/2$; set $\gamma \equiv \min(\gamma_1, \gamma_2)$. Monotonicity gives $(f + g)(\gamma) \leq f(\gamma_1) + g(\gamma_2) < \delta$ □

**Lemma 2.5** (Composition). [lem:challenge-composition] *Let $f, g$ be convergence challenge functions, and suppose $f$ is monotone increasing (i.e. $x \leq y \implies f(x) \leq f(y)$). Then $f \circ g$ is a convergence challenge function*

*Proof.* Positivity: $f(g(\gamma)) > 0$ since $g(\gamma) > 0$ and $f$ preserves positivity. For $\delta > 0$, pick $\eta > 0$ with $f(\eta) < \delta$, then $\gamma > 0$ with $g(\gamma) < \eta$. Monotonicity of $f$ gives $(f \circ g)(\gamma) \leq f(\eta) < \delta$ □

**Lemma 2.6** (Reciprocal). [lem:reciprocal-is-challenge] *The function $g : \mathbb{R}_+ \to \mathbb{R}_+ \equiv t \mapsto 1/t$ is a convergence challenge function*

*Proof.* For $t > 0$, $g(t) = 1/t > 0$. Given $\delta > 0$, choose $t \equiv 2/\delta > 0$, so $g(t) = \delta/2 < \delta$ □

**Theorem 2.2** (Closed Form of Convergence). [thm:convergence-closed-form] *Let $F$ be an ordered field, let $a : \mathbb{N}_1 \to F$. Then*

$$\left(\exists \ell \in F,\, \forall \varepsilon \in F_+,\, \exists N \in \mathbb{N}_1,\, \forall n \in \mathbb{N}_1,\, n \geq N,\, d(a_n, \ell) < \varepsilon\right)$$

$$\iff$$

$$\left(\exists \ell \in F,\, \forall \varepsilon \in F_+,\, \exists N \in \mathbb{N}_1,\, \forall n \in \mathbb{N}_1,\, n \geq N,\, d(a_n, \ell) \leq \varepsilon\right)$$

*Proof.* ($\Rightarrow$) If $d(a_n, \ell) < \varepsilon$, then $d(a_n, \ell) \leq \varepsilon$
($\Leftarrow$) Fix the witness $\ell \in F$. For any $\varepsilon \in F_+$, apply the right-hand condition at $\varepsilon/2 > 0$ to obtain $N \in \mathbb{N}_1$ such that

$$n \geq N \implies d(a_n, \ell) \leq \varepsilon/2 < \varepsilon$$

□

# 3 Limit of a Sequence

**Definition 3.1** (Limit of a Sequence)**.** [def:limit] Let $F$ be an ordered field. We say the sequence $a : \mathbb{N}_1 \to F$ *has a limit of* $\ell \in F$ if

$$\forall \varepsilon \in F_+,\ \exists N \in \mathbb{N}_1,\ \forall n \in \mathbb{N}_1,\ n \geq N,\ d(a_n, \ell) < \varepsilon$$

*Redirect (Converges To)* [def:converges-to]
*Moved to definition 3.1.*

**Definition 3.3** (Set of Convergent Sequences)**.** [def:convergent-sequences-set] Let $F$ be an ordered field. The *set of convergent sequences in* $F$ is

$$\mathrm{Conv}(F) \equiv \{a : \mathbb{N}_1 \to F \mid a \text{ is convergent}\}$$

LaTeX: \Conv (F)

**Definition 3.4** (Limit Operator)**.** [def:limit-op] Let $F$ be an ordered field. The *limit operator* is the function

$$\lim : \mathrm{Conv}(F) \to F \equiv a \mapsto \text{the unique } \ell \in F \text{ such that } a\text{has a limit of}\ell$$

We write

$$\lim(a) = \ell$$

The input is the whole sequence, rather than an individual term

LaTeX: \seqlim (a)

*Redirect (Limit of a Sequence)* [def:limit-of-sequence]
*Moved to definition 3.4.*

*Remark* 3.1 (Well-Definedness of the Limit)*.* [rem:limit-well-defined] The function lim in definition 3.4 is well-defined for two reasons:

- *Every input has an output*: for every $a \in \mathrm{Conv}(F)$, the definition of convergent sequence guarantees some $\ell \in F$ with $a$has a limit of$\ell$
- *Every input has exactly one output*: by theorem 1.1, this $\ell$ is unique

**Theorem 3.1** (Addition of the Limit)**.** [thm:limit-addition] *Let $F$ be an ordered field, and let $a, b \in \mathrm{Conv}(F)$. Then*

$$\lim(n \mapsto a_n + b_n) = \lim(a) + \lim(b)$$

*Proof.* We want to prove

$$\forall \varepsilon \in F_+,\ \exists N \in \mathbb{N}_1,\ \forall n \in \mathbb{N}_1,\ n \geq N \implies$$
$$|a_n + b_n - \lim(a) - \lim(b)| < \varepsilon$$

According to the definition of a limit, for every $\varepsilon \in F_+$, there exist $N_a, N_b \in \mathbb{N}_1$ such that, setting $N_M \equiv \max(N_a, N_b)$, both estimates hold for $n \geq N_M$:

$$|a_n - \lim(a)| < \varepsilon \quad \wedge \quad |b_n - \lim(b)| < \varepsilon$$

Therefore, by the triangle inequality for distance,

$$\begin{aligned} |a_n + b_n - \lim(a) - \lim(b)| &\leq |a_n - \lim(a)| + |b_n - \lim(b)| \\ &< \varepsilon + \varepsilon = 2 \cdot \varepsilon \end{aligned}$$

For any $\delta \in F_+$, apply the preceding estimate with $\varepsilon = \delta/2 > 0$. The error is then less than $2 \cdot (\delta/2) = \delta$, so

$$\lim(n \mapsto a_n + b_n) = \lim(a) + \lim(b)$$

□

**Theorem 3.2** (Multiplication of the Limit). [thm:limit-multiplication] *Let $F$ be an ordered field, and let $a, b \in \mathrm{Conv}(F)$. Then*

$$\lim(n \mapsto a_n \cdot b_n) = \lim(a) \cdot \lim(b)$$

*Proof.* For $n \mapsto a_n \cdot b_n$, we want to show that $\lim(a) \cdot \lim(b)$ is its limit, ie,

$$\begin{gathered}\forall \varepsilon \in F_+,\, \exists N \in \mathbb{N}_1,\, \forall n \in \mathbb{N}_1,\, n \geq N \implies \\ |a_n \cdot b_n - \lim(a) \cdot \lim(b)| < \varepsilon\end{gathered}$$

According to the definition of a limit, for every $\varepsilon \in F_+$, there exist $N_a, N_b \in \mathbb{N}_1$ such that

$$\begin{aligned} n \geq N_a &\implies |a_n - \lim(a)| < \varepsilon, \\ n \geq N_b &\implies |b_n - \lim(b)| < \varepsilon \end{aligned}$$

Let $N_M \equiv \max(N_a, N_b)$. Then, for every $n \geq N_M$, both inequalities hold
Adding and subtracting $\lim(a) \cdot b_n$, we have

$$\begin{aligned} &|a_n \cdot b_n - \lim(a) \cdot \lim(b)| \\ &\quad = |a_n \cdot b_n - \lim(a) \cdot b_n + \lim(a) \cdot b_n - \lim(a) \cdot \lim(b)| \\ &\quad \leq |a_n \cdot b_n - \lim(a) \cdot b_n| + |\lim(a) \cdot b_n - \lim(a) \cdot \lim(b)| \\ &\quad = |b_n| \cdot |a_n - \lim(a)| + |\lim(a)| \cdot |b_n - \lim(b)| \end{aligned}$$

The inequality uses the triangle inequality for distance
Therefore, when $n \geq N_M$,

$$\begin{aligned} |a_n \cdot b_n - \lim(a) \cdot \lim(b)| &\leq |b_n| \cdot \varepsilon + |\lim(a)| \cdot \varepsilon \\ &= \varepsilon \cdot \big(|b_n| + |\lim(a)|\big) \end{aligned}$$

Since all convergent sequences are bounded, choose a strict bound $M' \in F$ such that $\forall n \in \mathbb{N}_1$, $|b_n| < M'$, and let $M \equiv \max(M', 1)$. This bound is independent of $\varepsilon$. Hence $M \geq 1$ and

$$|a_n \cdot b_n - \lim(a) \cdot \lim(b)| < \varepsilon \cdot \big(M + |\lim(a)|\big)$$

For any $\delta \in F_+$, apply the preceding estimate with

$$\varepsilon = \frac{\delta}{M + |\lim(a)|} > 0$$

Since $M$ is independent of the tolerance, the error is then less than $\delta$. Therefore,

$$\lim(n \mapsto a_n \cdot b_n) = \lim(a) \cdot \lim(b)$$

□

**Theorem 3.3** (Scalar Multiplication of the Limit). [thm:limit-scalar-multiplication] *Let $F$ be an ordered field, let $a \in \mathrm{Conv}(F)$, and let $c \in F$. Then*

$$\lim(n \mapsto c \cdot a_n) = c \cdot \lim(a)$$

*Proof.* Define the constant sequence

$$p : \mathbb{N}_1 \to F \equiv n \mapsto c$$

For every $\varepsilon \in F_+$ and every $n \in \mathbb{N}_1$,

$$|p_n - c| = 0 < \varepsilon$$

Hence $p \in \mathrm{Conv}(F)$ and $\lim(p) = c$
By the definition of $p$,

$$\forall n \in \mathbb{N}_1, \quad c \cdot a_n = p_n \cdot a_n$$

By multiplication of limits, the sequence $n \mapsto p_n \cdot a_n$ is convergent, and

$$\begin{aligned} \lim(n \mapsto c \cdot a_n) &= \lim(n \mapsto p_n \cdot a_n) \\ &= \lim(p) \cdot \lim(a) \\ &= c \cdot \lim(a) \end{aligned}$$

□

**Theorem 3.4** (Linearity of the Limit). [thm:limit-linearity] *Let $F$ be an ordered field, let $a, b \in \mathrm{Conv}(F)$, and let $c, d \in F$. Then*

$$\lim(n \mapsto c \cdot a_n + d \cdot b_n) = c \cdot \lim(a) + d \cdot \lim(b)$$

*Proof.* This follows from addition of limits and scalar multiplication of limits □

**Lemma 3.1** (Reciprocal of the Limit). [lem:limit-reciprocal] *Let $F$ be an ordered field, and let $b \in \mathrm{Conv}(F)$ with $\forall n \in \mathbb{N}_1$, $b_n \neq 0$ and $\lim(b) \neq 0$. Then*

$$\lim\left(n \mapsto \frac{1}{b_n}\right) = \frac{1}{\lim(b)}$$

*Proof.* Define

$$p : \mathbb{N}_1 \to F \equiv n \mapsto \frac{1}{b_n}$$

We check whether $1/\lim(b)$ is a limit of $p$
For $n \in \mathbb{N}_1$,

$$\begin{aligned} \left| p_n - \frac{1}{\lim(b)} \right| &= \left| \frac{1}{b_n} - \frac{1}{\lim(b)} \right| \\ &= \left| \frac{\lim(b) - b_n}{b_n \cdot \lim(b)} \right| \\ &= \frac{|\lim(b) - b_n|}{|b_n \cdot \lim(b)|} \end{aligned}$$

To bound the denominator below, set $M \equiv |\lim(b)|/2 > 0$. By the convergence of $b$, choose $N_0 \in \mathbb{N}_1$ such that $n \geq N_0 \implies |b_n - \lim(b)| < M$. For $n \geq N_0$, the triangle inequality gives

$$|\lim(b)| \leq |\lim(b) - b_n| + |b_n| < M + |b_n|$$

Since $|\lim(b)| = 2 \cdot M$, we have $|b_n| > M$. So

$$\exists M \in F_+, \exists N_0 \in \mathbb{N}_1, \forall n \in \mathbb{N}_1, n \geq N_0 \implies |b_n| > M$$

Fix $\varepsilon \in F_+$ and choose $N_b \in \mathbb{N}_1$ so that $n \geq N_b \implies |b_n - \lim(b)| < \varepsilon$. Set $N \equiv \max(N_0, N_b)$.
For $n \geq N$,

$$|b_n \cdot \lim(b)| > M \cdot |\lim(b)|, \qquad |\lim(b) - b_n| < \varepsilon$$

Therefore,

$$\frac{|\lim(b) - b_n|}{|b_n \cdot \lim(b)|} < \frac{\varepsilon}{M \cdot |\lim(b)|}$$

For any $\delta \in F_+$, apply this estimate with

$$\varepsilon = \delta \cdot M \cdot |\lim(b)| > 0$$

The error is then less than $\delta$, so $p \in \mathrm{Conv}(F)$ and

$$\lim(p) = \frac{1}{\lim(b)}$$

□

**Theorem 3.5** (Division of the Limit). [thm:limit-division] *Let $F$ be an ordered field, and let $a, b \in \mathrm{Conv}(F)$ with $\forall n \in \mathbb{N}_1, b_n \neq 0$ and $\lim(b) \neq 0$. Then*

$$\lim\left(n \mapsto \frac{a_n}{b_n}\right) = \frac{\lim(a)}{\lim(b)}$$

*Proof.* Define

$$p : \mathbb{N}_1 \to F \equiv n \mapsto \frac{1}{b_n}$$

By the reciprocal lemma, $p \in \mathrm{Conv}(F)$ and $\lim(p) = 1/\lim(b)$. Since $a_n/b_n = a_n \cdot p_n$ for every $n \in \mathbb{N}_1$, multiplication of limits gives

$$\begin{aligned}\lim\left(n \mapsto \frac{a_n}{b_n}\right) &= \lim(n \mapsto a_n \cdot p_n) \\ &= \lim(a) \cdot \lim(p) \\ &= \frac{\lim(a)}{\lim(b)}\end{aligned}$$

□

**Theorem 3.6** (Order Preservation of Limits). [thm:limit-order-preservation] *Let $F$ be an ordered field, and let $a, b \in \mathrm{Conv}(F)$ with $\forall n \in \mathbb{N}_1$, $a_n \leq b_n$. Then*

$$\lim(a) \leq \lim(b)$$

**Lemma 3.2** (Shifting a Convergent Sequence). [lem:limit-index-shift] *Let $a : \mathrm{Conv}(F)$, where $F$ is an ordered field, and let $r \in \mathbb{N}$. Then $n \mapsto a(n + r)$ is convergent, and*

$$\lim(n \mapsto a(n + r)) = \lim(a)$$

*Proof.* For any $\varepsilon > 0$, choose $N$ for $a$. If $n \geq N$, then $n + r \geq N$, so $|a(n + r) - \lim(a)| < \varepsilon$ □

**Theorem 3.7** (Absolute Values of Limits). [thm:limit-absolute-value] *Let $a : \mathrm{Conv}(F)$, where $F$ is an ordered field. Then*

$$\lim(n \mapsto |a(n)|) = |\lim(a)|$$

*Proof.* The triangle inequality in both directions gives

$$||a(n)| - |\lim(a)|| \leq |a(n) - \lim(a)|$$

Thus the same $N$ works in the definition of the limit □

**Theorem 3.8** (Maxima of Limits). [thm:limit-maximum] *Let $a, b : \mathrm{Conv}(F)$, where $F$ is an ordered field. Then*

$$\lim(n \mapsto \max(a(n), b(n))) = \max(\lim(a), \lim(b))$$

*Proof.* Use $\max(x, y) = (x + y + |x - y|)/2$, linearity, and the absolute-value limit rule □

**Corollary 3.1** (Minima of Limits). [cor:limit-minimum] *Under the same assumptions,*

$$\lim(n \mapsto \min(a(n), b(n))) = \min(\lim(a), \lim(b))$$

*Proof.* Apply the maximum rule to $n \mapsto -a(n)$ and $n \mapsto -b(n)$, using $\min(x, y) = -\max(-x, -y)$ □

**Theorem 3.9** (Sequence Characterization of Limit Points). [thm:limit-point-sequence] *Let $(X, d)$ be a metric space, let $A \subseteq X$, and let $x \in X$. The point $x$ is a limit point of $A$ if and only if there exists a sequence $a : \mathbb{N}_1 \to A \setminus \{x\}$ converging to $x$, meaning*

$$\forall \varepsilon \in \mathbb{R}_+, \exists N \in \mathbb{N}_1, \forall n \in \mathbb{N}_1, \quad n \geq N \implies d(a_n, x) < \varepsilon$$

*Proof.* According to definition of limit, $x$ is a limit □

# 4 Bounded Sequences

**Definition 4.1** (Bounded Sequence). [def:bounded-sequence] Let $F$ be an ordered field. A sequence $a : \mathbb{N}_1 \to F$ is *bounded* if

$$\exists M \in F, \forall n \in \mathbb{N}_1, \quad d(a(n), 0) \leq M$$

**Definition 4.2** (Set of Bounded Sequences). [def:bounded-sequences-set] Let $F$ be an ordered field. The set of bounded sequences in $F$ is

$$\mathrm{Bdd}(F) \equiv \{a : \mathbb{N}_1 \to F \mid a \text{ is bounded}\}$$

`LaTeX: \Bdd (F)`

**Theorem 4.1** (Convergent Sequences are Bounded). [thm:convergent-sequences-are-bounded] *Let $F$ be an ordered field and let $a : \mathrm{Conv}(F)$. Then $a$ is bounded, so $a \in \mathrm{Bdd}(F)$*

*Proof.* Suppose the convergent sequence $a$, $\lim a = \ell$
Take $\varepsilon = 1$, then there exists $N \in \mathbb{N}_1$ such that for all $n \in \mathbb{N}_1$ with $n > N$, $d(a_n, \ell) < 1$
According to the triangle inequality for distance, and $d(a_n, \ell) < 1$, we have

$$d(0, a_n) \leq d(0, \ell) + d(\ell, a_n) < d(0, \ell) + 1$$

Which means for all $n \in \mathbb{N}_1$ with $n > N$, $d(0, a_n) < d(0, \ell) + 1$
And for $n \in \mathbb{N}_1$ with $n \leq N$, we have $d(0, a_n) \leq \max\{d(0, a_1), d(0, a_2), ..., d(0, a_N)\}$
So, we can take $M = \max\{\max\{d(0, a_1), d(0, a_2), ..., d(0, a_N)\}, d(0, \ell) + 1\}$. Hence $M$ exists for all convergent sequences $a$ □

*Remark* 4.1. Although Convergent $\implies$ Bounded, the converse is not true. A bounded sequence may not be convergent
For example, the sequence $a_n = (-1)^n$ is bounded (since $d(a_n, 0) = 1$ for all $n$), but not convergent

# 5 Monotone Convergence Theorem

**Definition 5.1** (Increasing Sequence). [def:increasing-sequence] A sequence $a : \mathbb{N}_1 \to F$ is said to be *increasing* if

$$\forall n \in \mathbb{N}_1, a_n \leq a_{n+1}$$

**Definition 5.2** (Strictly Increasing Sequence). [def:strictly-increasing-sequence] A sequence $a : \mathbb{N}_1 \to F$ is said to be *strictly increasing* if

$$\forall n \in \mathbb{N}_1, a_n < a_{n+1}$$

**Definition 5.3** (Decreasing Sequence). [def:decreasing-sequence] A sequence $a : \mathbb{N}_1 \to F$ is said to be *decreasing* if

$$\forall n \in \mathbb{N}_1, a_n \geq a_{n+1}$$

**Definition 5.4** (Strictly Decreasing Sequence). [def:strictly-decreasing-sequence] A sequence $a : \mathbb{N}_1 \to F$ is said to be *strictly decreasing* if

$$\forall n \in \mathbb{N}_1, a_n > a_{n+1}$$

**Lemma 5.1** (Strictly Increasing Extends to All Indices). [lem:strictly-increasing-extends] *Let $a : \mathbb{N}_1 \to F$ be strictly increasing. Then $\forall n, m \in \mathbb{N}_1,\ n > m \implies a_n > a_m$*

**Lemma 5.2** (Increasing Extends to All Indices). [lem:increasing-extends] *Let $a : \mathbb{N}_1 \to F$ be increasing. Then $\forall n, m \in \mathbb{N}_1,\ n \geq m \implies a_n \geq a_m$*

**Lemma 5.3** (Strictly Decreasing Extends to All Indices). [lem:strictly-decreasing-extends] *Let $a : \mathbb{N}_1 \to F$ be strictly decreasing. Then $\forall n, m \in \mathbb{N}_1,\ n > m \implies a_n < a_m$*

**Lemma 5.4** (Decreasing Extends to All Indices). [lem:decreasing-extends] *Let $a : \mathbb{N}_1 \to F$ be decreasing. Then $\forall n, m \in \mathbb{N}_1,\ n \geq m \implies a_n \leq a_m$*

**Definition 5.5** (Monotone Sequence). [def:monotone-sequence] A sequence $a : \mathbb{N}_1 \to F$ is said to be *monotone* if it is either increasing or decreasing

**Theorem 5.1** (Monotone Convergence Theorem). [thm:mct] *Let $F$ be a complete ordered field, and let $a : \mathrm{Bdd}(F)$ be a monotone sequence. Then $a$ is convergent*

*Proof.* Suppose $a$ is a bounded sequence in $F$ and increasing sequence, which means exist $x = \sup\{a_n : n \in \mathbb{N}_1\}$ (by the Axiom of Completeness, applied to the nonempty set $\{a_n : n \in \mathbb{N}_1\}$ bounded above in $F$), and $\forall n, m \in \mathbb{N}_1,\ n \leq m \implies a_n \leq a_m$
According to the $\varepsilon$ criterion for supremum, $\forall n \in \mathbb{N}_1,\ a_n \leq x$ and $\forall \varepsilon \in F_+,\ \exists N \in \mathbb{N}_1,\ a_N > x - \varepsilon$
Adding the increasing property, ie

$$N \leq n \implies a_N \leq a_n \leq x \implies (a_N > x - \varepsilon \implies a_n \geq a_N > x - \varepsilon)$$

We can have

$$\forall \varepsilon \in F_+,\ \exists N \in \mathbb{N}_1,\ \forall n \in \mathbb{N}_1 \geq N,\ a_n > x - \varepsilon$$

Also,

$$a_n \leq x \wedge \varepsilon > 0 \implies a_n < x + \varepsilon$$

$$a_n > x - \varepsilon \wedge a_n < x + \varepsilon \implies x - \varepsilon < a_n < x + \varepsilon \implies d(a_n, x) < \varepsilon$$

Notice that such $x$ is the supremum of the upper bounded sequence $a$, it exists.

$$\exists x \in F,\ \forall \varepsilon \in F_+,\ \exists N \in \mathbb{N}_1,\ \forall n \in \mathbb{N}_1 \geq N,\ d(a_n, x) < \varepsilon$$

Which means it is a convergent sequence and its limit is $x$
If instead $a$ is decreasing, the same argument applies with $x \equiv \inf\{a_n : n \in \mathbb{N}_1\}$ (which exists by the Axiom of Completeness, since $\{a_n : n \in \mathbb{N}_1\}$ is nonempty and bounded below), using the dual $\varepsilon$-criterion for the infimum $\square$

# 6 The Number $e$

**Definition 6.1** (Defining Sequence for $e$). [def:e-defining-sequence]

$$u : \mathbb{N}_1 \to \mathbb{R} \equiv n \mapsto \left(1 + \frac{1}{n}\right)^n$$

**Theorem 6.1** (Convergence of the Defining Sequence for $e$). [thm:e-sequence-convergence]

**Definition 6.2** (The Number $e$). [def:euler-number]

$$e : \mathbb{R} \equiv \lim\big(u\big)$$

# 7 Comparison of Sequences

**Theorem 7.1** (Comparison for Monotone Sequences). [thm:monotone-comparison-increasing] *Let $F$ be a complete ordered field, and let $s, t : \mathbb{N}_1 \to F$ be increasing with $s_n \leq t_n$ for all $n \in \mathbb{N}_1$. If $t$ is convergent, then*

$$s \text{ is convergent and } \lim(s) \leq \lim(t)$$

*Proof.* **TODO** □

**Theorem 7.2** (Comparison for Decreasing Sequences). [thm:monotone-comparison-decreasing] *Let $F$ be a complete ordered field, and let $s, t : \mathbb{N}_1 \to F$ be decreasing with $s_n \leq t_n$ for all $n \in \mathbb{N}_1$. If $s$ is convergent, then*

$$t \text{ is convergent and } \lim(s) \leq \lim(t)$$

*Proof.* Apply theorem 7.1 to the increasing sequences $n \mapsto -t_n$ and $n \mapsto -s_n$ □

# 8 Squeeze Theorem

**Theorem 8.1** (Squeeze Theorem). [thm:squeeze-theorem] *Let $F$ be an ordered field, and let $a, x, b : \mathbb{N}_1 \to F$ with $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}_1$. If $a$ and $b$ are convergent with $\lim(a) = \lim(b) = L$, then $x$ is convergent and $\lim(x) = L$*

*Proof.* **TODO** □