Lecture 3: Metric Spaces and Topology

Mathematical Analysis

1 Metric Spaces

Definition 1.1 (Metric). [def:metric] Let X be a set. A *metric*, or *distance function*, on X is a function $d : X \times X \to \mathbb{R}$ satisfying the following four axioms: nonnegativity, identity of indiscernibles, symmetry, and the triangle inequality

Axiom 1.1 (Nonnegativity). [axm:metric-nonnegativity] *If d is a metric on X, then*

$$\forall x, y \in X, \quad d(x, y) \geq 0$$

Axiom 1.2 (Identity of Indiscernibles). [axm:metric-separation] *If d is a metric on X, then*

$$\forall x, y \in X, \quad d(x, y) = 0 \iff x = y$$

Axiom 1.3 (Symmetry). [axm:metric-symmetry] *If d is a metric on X, then*

$$\forall x, y \in X, \quad d(x, y) = d(y, x)$$

Axiom 1.4 (Triangle Inequality). [axm:metric-triangle-inequality] *If d is a metric on X, then*

$$\forall x, y, z \in X, \quad d(x, z) \leq d(x, y) + d(y, z)$$

Definition 1.2 (Metric Space). [def:metric-space] A *metric space* is a pair (X, d), where X is a set and d is a metric on X

2 Open Balls and Neighbourhoods

Definition 2.1 (Open Balls). [def:open-balls] Let (X, d) be a metric space. The *open-ball generator* is

$$\text{OpenBall} : \mathbb{R}_+ \to \big(X \to \mathcal{P}(X)\big) \equiv r \mapsto x \mapsto \{y \in X : d(x, y) < r\}$$

For $r \in \mathbb{R}_+$ and $x \in X$, $\text{OpenBall}(r)(x)$ is the *open ball centred at x with radius r*

`LaTeX: \OpenBall (r)(x)`

Remark 2.1 (Open Ball Intuition). [rem:open-ball-intuition] An open ball in a metric space (X, d), centred at $x \in X$ with positive radius r, is the collection of all elements of X whose distance from x is less than r

Convention 2.1 (Open Ball Notation). [conv:open-ball-notation] For the open-ball generator, write

$$B_r(x) \equiv \text{OpenBall}(r)(x)$$

`LaTeX: \ball {r}{x}`

Definition 2.2 (Neighbourhood). [def:neighbourhood] Let (X, d) be a metric space, and let $x \in X$. A set $U \subseteq X$ is a *neighbourhood of x* if

$$\exists r \in \mathbb{R}_+, \quad B_r(x) \subseteq U$$

Definition 2.3 (Bounded Set in a Metric Space). [def:metric-bounded-set] Let (X, d) be a metric space. A subset $S \subseteq X$ is *bounded* if

$$\exists M \in \mathbb{R}_+, \forall x, y \in S, \quad d(x, y) < M$$

Convention 2.2 (Radius Neighbourhood Notation). [conv:radius-neighbourhood-notation] In a metric space (X, d), some texts call the open ball $B_r(x)$, with $r \in \mathbb{R}_+$ and $x \in X$, the *r-neighbourhood of* x and write

$$V_r(x) \equiv B_r(x)$$

Lemma 2.1 (The Empty Set and the Whole Space). [lem:metric-empty-and-whole-open] *Let (X, d) be a metric space. Then*

$$\forall U \in \{\emptyset, X\},\ \forall x \in U,\ \exists r \in \mathbb{R}_+, \quad B_r(x) \subseteq U$$

Proof. **Case $U = X$:** For every $x \in X$, choose $r = 1$. By definition, $B_1(x) \subseteq X$
Case $U = \emptyset$: There is no $x \in \emptyset$, so the universally quantified condition holds vacuously
The empty-set case does not require finding a ball contained in $\emptyset$, since there is no centre to consider □

Lemma 2.2 (Finite Intersections). [lem:metric-finite-intersections] *Let (X, d) be a metric space, and let $U, V \subseteq X$ satisfy*

$$\forall x \in U,\ \exists r \in \mathbb{R}_+, \quad B_r(x) \subseteq U$$
$$\forall x \in V,\ \exists r \in \mathbb{R}_+, \quad B_r(x) \subseteq V$$

Then

$$\forall x \in U \cap V,\ \exists r \in \mathbb{R}_+, \quad B_r(x) \subseteq U \cap V$$

Proof. Let $x \in U \cap V$. By the assumptions, choose $r_U, r_V \in \mathbb{R}_+$ such that $B_{r_U}(x) \subseteq U$ and $B_{r_V}(x) \subseteq V$. Let $r = \min(r_U, r_V) > 0$. Then

$$B_r(x) \subseteq B_{r_U}(x) \subseteq U, \qquad B_r(x) \subseteq B_{r_V}(x) \subseteq V$$

Therefore, $B_r(x) \subseteq U \cap V$ □

Lemma 2.3 (Arbitrary Unions). [lem:metric-arbitrary-unions] *Let (X, d) be a metric space, and let $U : I \to \mathcal{P}(X)$ satisfy*

$$\forall i \in I,\ \forall x \in U(i),\ \exists r \in \mathbb{R}_+, \quad B_r(x) \subseteq U(i)$$

Then

$$\forall x \in \bigcup_I U,\ \exists r \in \mathbb{R}_+, \quad B_r(x) \subseteq \bigcup_I U$$

Proof. Let $x \in \bigcup_I U$. Choose $i \in I$ with $x \in U(i)$. By the assumption, choose $r \in \mathbb{R}_+$ such that

$$B_r(x) \subseteq U(i) \subseteq \bigcup_I U$$

□

3 Distance on an Ordered Field

Definition 3.1 (Distance). [def:distance] Let F be an ordered field. For $x, y \in F$, the *distance* between x and y is

$$d(x, y) \equiv |x - y|$$

`LaTeX: \dist {x}{y}`

Lemma 3.1 (Symmetry of Distance). [lem:distance-symmetry] *Let F be an ordered field. For all $x, y \in F$,*

$$d(x, y) = d(y, x)$$

Proof. By definition, $d(x,y) = |x-y|$ and $d(y,x) = |y-x| = |-(x-y)|$. It suffices to show $|-(x-y)| = |x-y|$; we split on the sign of $x-y$
Case 1: $x-y \geq 0$. Then $|x-y| = x-y$. Also $-(x-y) \leq 0$, so $|-(x-y)| = -(-(x-y)) = x-y = |x-y|$
Case 2: $x-y < 0$. Then $|x-y| = -(x-y)$. Also $-(x-y) > 0$, so $|-(x-y)| = -(x-y) = |x-y|$ □

Lemma 3.2 (Triangle Inequality for Distance). [lem:triangle-inequality-distance] *Let F be an ordered field. For all $x, y, z \in F$,*

$$d(x,z) \leq d(x,y) + d(y,z)$$

Proof. Let $a = x-y$ and $b = y-z$. Then $x-z = a+b$, so $d(x,z) = |a+b|$. We split on the sign of $a+b$
Case 1: $a+b \geq 0$. Then $|a+b| = a+b$. Since $a \leq |a|$ and $b \leq |b|$,

$$|a+b| = a+b \leq |a| + |b| = d(x,y) + d(y,z)$$

Case 2: $a+b < 0$. Then $|a+b| = -(a+b) = -a-b$. Since $-a \leq |a|$ and $-b \leq |b|$,

$$|a+b| = -a-b \leq |a| + |b| = d(x,y) + d(y,z)$$

□

Corollary 3.1 (Metric on a Complete Ordered Field). [cor:complete-ordered-field-metric] *Let F be a complete ordered field, and let d be the absolute-value distance on F. Under the identification with the reals, (F,d) is a metric space*

Proof. Under this identification, the distance is real-valued. By the definition of absolute value,

$$d(x,y) = |x-y| \geq 0, \qquad d(x,y) = 0 \iff x-y = 0 \iff x = y$$

Symmetry and the triangle inequality give the remaining two metric axioms □

Lemma 3.3 (ε-Criterion for Equality). [lem:epsilon-equality] *Let F be an ordered field. For $a, b \in F$,*

$$a = b \iff \forall \varepsilon \in F_+,\ |a-b| < \varepsilon$$

Proof. In order to proof Epsilon Equality Lemma, we need to prove both directions
(a): $a = b \implies \forall \varepsilon \in F_+, |a-b| < \varepsilon$
Suppose $a = b$. Then $|a-b| = 0$. For any $\varepsilon > 0$, we have $|a-b| = 0 < \varepsilon$
(b): $\forall \varepsilon \in F_+, |a-b| < \varepsilon \implies a = b$
Assume $a \neq b$
Then, this means that

$$\exists \varepsilon_0 \in F_+, |a-b| = \varepsilon_0$$

Taking $\varepsilon = \varepsilon_0$ gives $\varepsilon_0 = |a-b| < \varepsilon_0$, a contradiction
So, we have $\forall \varepsilon \in F_+, |a-b| < \varepsilon \implies \neg a \neq b$
Since $\neg a \neq b \iff a = b$, So, $\forall \varepsilon \in F_+, |a-b| < \varepsilon \implies a = b$
Therefore, both implications hold, and hence proven □

4 Topological Spaces

Definition 4.1 (Topology)**.** [def:topology] Let X be a set. A *topology* on X, called $\mathcal{T}$, is a subset of $\mathcal{P}X$ satisfying containment of the empty set and the whole set, closure under finite intersections, and closure under arbitrary unions

Axiom 4.1. [axm:topology-empty-and-whole] *If $\mathcal{T}$ is a topology on X, then*

$$\emptyset, X \in \mathcal{T}$$

Axiom 4.2. [axm:topology-finite-intersections] *If $U, V \in \mathcal{T}$, then*

$$U \cap V \in \mathcal{T}$$

Axiom 4.3. [axm:topology-arbitrary-unions] *If $U : I \to \mathcal{T}$ is a family of sets in $\mathcal{T}$, then*

$$\bigcup_I U \in \mathcal{T}$$

Definition 4.2 (Topological Spaces)**.** [def:topological-space] A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X

Definition 4.3 (Induced Topology)**.** [def:induced-topology] Let (X, d) be a metric space, where we define a *induced topology* $\mathcal{T}_d$ induced by d ;s

$$\mathcal{T}_d \equiv \{U \in \mathcal{P}(X) \mid \forall x \in U,\, \exists r \in \mathbb{R}_+,\, B_r(x) \subseteq U\}$$

Theorem 4.1 (Induced Topology is a Topology)**.** [thm:induced-topology-is-a-topology]

$$(X, d) \text{ is a metric space} \implies (X, \mathcal{T}_d) \text{ is a topological space}$$

Proof. By definition, $\mathcal{T}_d \subseteq \mathcal{P}(X)$. The lemmas on the empty set and the whole space, finite intersections, and arbitrary unions give

$$\emptyset, X \in \mathcal{T}_d, \qquad U, V \in \mathcal{T}_d \implies U \cap V \in \mathcal{T}_d, \qquad U : I \to \mathcal{T}_d \implies \bigcup_I U \in \mathcal{T}_d$$

Thus $\mathcal{T}_d$ satisfies the three topology axioms, so $(X, \mathcal{T}_d)$ is a topological space □

5 Open and Closed Sets

Definition 5.1 (Open Sets)**.** [def:open-sets] A set $U \subseteq X$ is *open* in a topological space $(X, \mathcal{T})$ if $U \in \mathcal{T}$.

Definition 5.2 (Closed Sets)**.** [def:closed-sets] A set $C \subseteq X$ is *closed* in a topological space $(X, \mathcal{T})$ if its complement $X \setminus C$ is open, i.e., $X \setminus C \in \mathcal{T}$.

Corollary 5.1 (Arbitrary Unions of Open Sets)**.** [cor:open-arbitrary-unions] *Let $(X, \mathcal{T})$ be a topological space, and let $U : I \to \mathcal{P}(X)$. If every $U(i)$ is open, then*

$$\bigcup(U) \in \mathcal{T}$$

Proof. Each $U(i) \in \mathcal{T}$ by the definition of an open set, so the arbitrary-union axiom gives $\bigcup(U) \in \mathcal{T}$ □

Corollary 5.2 (Finite Intersections of Open Sets). [cor:open-finite-intersections] *Let $(X, \mathcal{T})$ be a topological space, and let $U : I \to \mathcal{P}(X)$, where I is finite and nonempty. If every $U(i)$ is open, then*

$$\bigcap(U) \in \mathcal{T}$$

Proof. A single member belongs to $\mathcal{T}$. Applying the intersection axiom repeatedly proves the claim by induction on the number of sets □

Corollary 5.3 (Arbitrary Intersections of Closed Sets). [cor:closed-arbitrary-intersections] *In a topological space $(X, \mathcal{T})$, let $C : I \to \mathcal{P}(X)$, where I is nonempty. If every $C(i)$ is closed, then $\bigcap(C)$ is closed*

Proof. Each $X \setminus C(i)$ is open by the definition of a closed set. By De Morgan's laws and the open-union corollary,

$$X \setminus \bigcap(C) = \bigcup(i \mapsto X \setminus C(i)) \in \mathcal{T}$$

Thus $\bigcap(C)$ is closed □

Corollary 5.4 (Finite Unions of Closed Sets). [cor:closed-finite-unions] *Let $(X, \mathcal{T})$ be a topological space, and let $C : I \to \mathcal{P}(X)$, where I is finite. If every $C(i)$ is closed, then $\bigcup(C)$ is closed*

Proof. If I is empty, then $\bigcup(C) = \emptyset$ is closed since $X \in \mathcal{T}$. Otherwise, each $X \setminus C(i)$ is open. By De Morgan's laws and the open-intersection corollary,

$$X \setminus \bigcup(C) = \bigcap(i \mapsto X \setminus C(i)) \in \mathcal{T}$$

Thus $\bigcup(C)$ is closed □

Proposition 5.1 (Open Balls are Open). [prop:metric-open-balls-open] *In a metric space (X, d), every open ball is open in the induced topology*

Proof. Let $z \in B_r(x)$, and choose $s = r - d(x, z) > 0$. For $y \in B_s(z)$, the triangle inequality gives

$$d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + s = r$$

Thus $B_s(z) \subseteq B_r(x)$, so $B_r(x)$ is open □

6 Limit Points and Isolated Points

Definition 6.1 (Limit Points). [def:limit-points] Let (X, d) be a metric space. A *limit point* $x \in X$ of a set $U \subseteq X$ is a point such that

$$\forall \varepsilon \in \mathbb{R}_+, \quad B_\varepsilon(x) \cap \big(U \setminus \{x\}\big) \neq \emptyset$$

Remark 6.1 (Limit Point Intuition). [rem:limit-point-intuition] A limit point may lie inside or outside the set: no matter how small a ball centred at it is, that ball contains a point of the set different from the centre itself

Lemma 6.1 (Balls Disjoint from a Set). [lem:ball-disjoint-limit-points] *Let (X, d) be a metric space, let $A \subseteq X$, and let L be the set of limit points of A. For $x \in X$ and $r \in \mathbb{R}_+$,*

$$B_r(x) \cap A = \emptyset \implies B_r(x) \cap L = \emptyset$$

Proof. If point ℓ is a limit point of A, then

$$\forall \varepsilon \in \mathbb{R}_+,\ B_\varepsilon(\ell) \cap (A \setminus \{\ell\}) \neq \emptyset$$

Since $B_r(x) \cap A = \emptyset$, Assume such ℓ exist in $B_r(x)$, then $\ell \in B_r(x)$, which means

$$d(\ell, x) < r$$

Then, choose $\varepsilon = r - d(\ell, x) > 0$. For any $y \in B_\varepsilon(\ell)$, the triangle inequality and symmetry give

$$d(x, y) \leq d(x, \ell) + d(\ell, y) < d(\ell, x) + \varepsilon = r$$

Thus we have $B_\varepsilon(\ell) \subseteq B_r(x)$
However we know that $B_r(x) \cap A = \emptyset$, which means $B_\varepsilon(\ell) \cap A = \emptyset$
Which creates an contradiction. So, there does not exist a limit point ℓ in $B_r(x)$, which means $B_r(x) \cap L = \emptyset$ □

Redirect (Sequence Characterization of Limit Points) [thm:limit-point-sequence]
Moved to theorem limit point sequence.

Definition 6.2 (Isolated Points)**.** [def:isolated-points] Let (X, d) be a metric space. An *isolated point* $x \in X$ of a set $U \subseteq X$ is a point such that

$$\exists \varepsilon \in \mathbb{R}_+, \quad B_\varepsilon(x) \cap U = \{x\}$$

Theorem 6.2 (Isolated or Limit Point)**.** [thm:point-isolated-or-limit] *Let (X, d) be a metric space, let $A \subseteq X$, and let $x \in A$. Then x is either an isolated point or a limit point of A, but not both*

Proof. **TODO** □

7 Closure

Definition 7.1 (Closure)**.** [def:closure] Let $(X, \mathcal{T})$ be a topological space. The *closure function* $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ assigns to each $A \subseteq X$ the smallest closed subset of X containing A. For $A, C \subseteq X$,

$$\mathrm{cl}(A) = C \iff \begin{cases} A \subseteq C, & C \text{ is closed} \\ \forall D \subseteq X, & (A \subseteq D \text{ and } D \text{ is closed}) \implies C \subseteq D \end{cases}$$

`LaTeX: \closure (A)`

Pf. Well-definedness of closure. **TODO** □

Convention 7.1 (Closure Notation)**.** [conv:closure-notation] For the closure function, write

$$\overline{A} \equiv \mathrm{cl}(A)$$

`LaTeX: \cl {A}`

Lemma 7.1 (Closed Supersets Contain Limit Points)**.** [lem:closure-from-limit-points] *Let (X, d) be a metric space, let $A \subseteq X$, and let L be the set of limit points of A. Let $D \subseteq X$ be closed in the induced topology. Then*

$$A \subseteq D \implies L \subseteq D$$

Proof. Let D be a closed set containing A. Which means $E \equiv X \setminus D$ is open.
Since $\ell \in L$ is a limit point of A, we know that

$$\forall \varepsilon \in \mathbb{R}_+,\ B_\varepsilon(\ell) \cap A \setminus \{\ell\} \neq \emptyset$$

. Suppose $\exists \ell \in L,\ \ell \notin D,\ \ell \notin A$. This means $\ell \in E$.
Since all element which is in A not in E, and since for such ℓ we already said

$$B_\varepsilon(\ell) \cap A \setminus \{\ell\} \neq \emptyset$$

Then,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in B_\varepsilon(\ell),\ \delta \in A \implies B_\varepsilon(\ell) \text{ is not a subset of } E$$

Which means that there does not exist a r such that $B_r(\ell) \subseteq E$, contradicts with the requirement where E must be an open set □

Lemma 7.2 (Adjoining Limit Points Gives a Closed Set)**.** [lem:adjoining-limit-points] *Let (X, d) be a metric space, let $A \subseteq X$, and let L be the set of limit points of A. Then $A \cup L$ is closed in the induced topology*

Proof. Define $C \equiv X \setminus (A \cup L)$. Choose $x \in C$. It must not be a limit point of A. Then,
By definition of limit point, we know that

$$\forall x \in C,\ \exists r \in \mathbb{R}_+,\ B_r(x) \setminus \{x\} \cap A = \emptyset$$

Since $x \notin A$, simplify,

$$\forall x \in C,\ \exists r \in \mathbb{R}_+,\ B_r(x) \cap A = \emptyset$$

By the ball lemma, $B_r(x) \cap L = \emptyset$, so $B_r(x) \subseteq C$. Since $x \in C$ was arbitrary, C is open by the definition of the induced topology. Hence $A \cup L$ is closed □

Theorem 7.1 (Closure by Adjoining Limit Points)**.** [thm:closure-limit-points] *Let (X, d) be a metric space, and let $A \subseteq X$. With closure taken in the induced topology, write*

$$L \equiv \{x \in X : x \text{ is a limit point of } A\}$$

Then

$$\overline{A} = A \cup L$$

Proof. By the lemma on adjoining limit points, $A \cup L$ is closed and contains A, so the definition of closure gives $\overline{A} \subseteq A \cup L$. Conversely, $\overline{A}$ is a closed set containing A, so the closed-superset lemma gives $L \subseteq \overline{A}$. Hence $A \cup L \subseteq \overline{A}$, proving equality □

Remark 7.1 (Adding Limit Points Once is Enough). [rem:closure-limit-points-intuition] Start with A and add all its limit points to obtain $A \cup L$. This set is already the closure of A, so it is closed and contains all its own limit points. No further points need to be added, so repeating the process leaves $A \cup L$ unchanged

8 Compact Sets

Definition 8.1 (Cover)**.** [def:cover] Let X be a set and $S \subseteq X$. A *cover* of S is a family $U : I \to \mathcal{P}(X)$ such that

$$S \subseteq \bigcup_I U$$

Definition 8.2 (Finite Cover). [def:finite-cover] A cover $U : I \to \mathcal{P}(X)$ of $S \subseteq X$ is *finite* if its index set I is finite, equivalently,

$$|I| < |\mathbb{N}|$$

Definition 8.3 (Open Cover). [def:open-cover] Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$. A cover $U : I \to \mathcal{P}(X)$ of S is an *open cover* if

$$\forall i \in I, \quad U(i) \in \mathcal{T}$$

Definition 8.4 (Subcover). [def:subcover] Let $U : I \to \mathcal{P}(X)$ be a cover of $S \subseteq X$. A *subcover* of U is a cover $V : J \to \mathcal{P}(X)$ of S such that

$$\forall j \in J, \exists i \in I, \quad V(j) = U(i)$$

The subcover is *finite* if V is a finite cover

Definition 8.5 (Compact Set). [def:compact-set] Let $(X, \mathcal{T})$ be a topological space. A subset $S \subseteq X$ is *compact* if every open cover of S has a finite subcover

9 The Usual Metric and Topology on $\mathbb{R}$

Definition 9.1 (Usual Topology on the Real Line). [def:usual-topology-real] Equip $\mathbb{R}$ with the absolute-value distance, which is a metric:

$$d(x, y) = |x - y|, \qquad x, y \in \mathbb{R}$$

The *usual topology* on $\mathbb{R}$ is the topology induced by the absolute-value metric d. In this section, balls and open sets in $\mathbb{R}$ refer to this metric and topology

Proposition 9.1 (Open Balls on the Real Line). [prop:real-open-balls] *For $x \in \mathbb{R}$ and $r \in \mathbb{R}_+$, the open ball is the open interval*

$$B_r(x) = (x - r, x + r)$$

Proof. For $y \in \mathbb{R}$,

$$y \in B_r(x) \iff |x - y| < r \iff x - r < y < x + r \iff y \in (x - r, x + r)$$

□

Corollary 9.1 (Open Intervals are Open). [cor:real-open-intervals] *For $a, b \in \mathbb{R}$ with $a < b$, the open interval (a, b) is open in the usual topology*

Proof. For each $x \in (a, b)$, we can choose a ball centred at x with radius $r = \min(x - a, b - x) > 0$. Then

$$B_r(x) = (x - r, x + r) \subseteq (a, b)$$

Thus every point of (a, b) has a ball contained in (a, b), so (a, b) is open by the definition of the induced topology □

Proposition 9.2 (Open Sets as Unions of Open Intervals). [prop:real-open-sets-intervals] *A set $U \subseteq \mathbb{R}$ is open in the usual topology if and only if it is a union of open intervals*

Proof. **(⇒)** Suppose U is open. Let

$$I \equiv \{(x, r) \in U \times \mathbb{R}_+ : B_r(x) \subseteq U\}, \qquad J : I \to \mathcal{P}(\mathbb{R}) \equiv (x, r) \mapsto (x - r, x + r)$$

Every $J(x, r) = B_r(x)$ is contained in U, and each $x \in U$ belongs to one such interval by the definition of $\mathcal{T}_d$. Hence $U = \bigcup(J)$
(⇐) Each open interval is open, and arbitrary unions of open sets are open □

Proposition 9.3 (Closed Intervals are Closed). [prop:real-closed-intervals] *For $a, b \in \mathbb{R}$ with $a \leq b$, the closed interval $[a, b]$ is closed in the usual topology*

Proof. Let $x \in \mathbb{R} \setminus [a, b]$. If $x < a$, choose $r = a - x > 0$; if $x > b$, choose $r = x - b > 0$. In either case, the ball centred at x satisfies

$$B_r(x) = (x - r, x + r) \subseteq \mathbb{R} \setminus [a, b]$$

Thus $\mathbb{R} \setminus [a, b]$ is open, so $[a, b]$ is closed by the definition of a closed set □

10 Heine-Borel Theorem

Lemma 10.1 (Compact Sets are Bounded). [lem:compact-implies-bounded] *Let (X, d) be a metric space. Every compact subset $S \subseteq X$, in the induced topology, is bounded*

Proof. **TODO** □

Lemma 10.2 (Compact Sets are Closed). [lem:compact-implies-closed] *Let (X, d) be a metric space. Every compact subset $S \subseteq X$, in the induced topology, is closed*

Proof. **TODO** □

Lemma 10.3 (Closed and Bounded Sets are Compact). [lem:real-closed-bounded-compact] *Every subset $S \subseteq \mathbb{R}$ that is closed in the usual topology and bounded is compact*

Proof. **TODO** □

Theorem 10.1 (Heine-Borel Theorem). [thm:heine-borel] *A subset $S \subseteq \mathbb{R}$ is compact in the usual topology if and only if it is closed and bounded*

Proof. **TODO** □

Corollary 10.1 (Compact Sets Attain Their Bounds). [cor:compact-real-extrema] *Let $S \subseteq \mathbb{R}$ be nonempty and compact in the usual topology. Its minimum and maximum exist, and*

$$\min(S) = \inf(S) \in S, \qquad \max(S) = \sup(S) \in S$$

Proof. **TODO** □

11 Cantor Sets

Definition 11.1 (Cantor Stages). [def:cantor-stages] The function $A : \mathbb{N} \to \mathcal{P}(\mathbb{R})$ is defined recursively by

$$A(0) \equiv [0, 1], \qquad A(n + 1) \equiv \left\{ \frac{x}{3} : x \in A(n) \right\} \cup \left\{ \frac{x + 2}{3} : x \in A(n) \right\}$$

Each stage removes the open middle third of every remaining interval, keeping its endpoints

Example 11.1 (The First Cantor Stages). [eg:cantor-first-stages]

$$A(1) = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$
$$A(2) = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$

Definition 11.2 (Middle-Third Cantor Set)**.** [def:cantor-set] The *middle-third Cantor set* is the intersection of all Cantor stages:

$$\mathcal{C} \equiv \bigcap(A)$$

Thus $x \in \mathcal{C}$ if and only if x belongs to every $A(n)$

Proposition 11.1 (Properties of the Cantor Stages)**.** [prop:cantor-stage-properties] *For each $n \in \mathbb{N}$, the stage $A(n)$ is a union of 2^n disjoint closed intervals of length 3^{-n}. Moreover,*

$$\{0, 1\} \subseteq A(n+1) \subseteq A(n) \subseteq [0, 1]$$

Proof. **TODO** □

Theorem 11.1 (The Cantor Set is Closed)**.** [thm:cantor-closed] *The Cantor set $\mathcal{C}$ is closed in $\mathbb{R}$ with the usual topology*

Proof. **TODO** □

Corollary 11.1 (The Cantor Set is Compact)**.** [cor:cantor-compact] *The Cantor set is a nonempty compact subset of $\mathbb{R}$*

Proof. **TODO** □