Lecture 2: Cardinality and Countability

Mathematical Analysis

1 Cardinality

Definition 1.1 (Cardinality). [def:cardinality] The *cardinality* of a set A, denoted by $|A|$, is the number of elements in A

LaTeX: \card {A}

Definition 1.2 (Cardinal Inequality). [def:cardinal-inequality] For sets A and B,

$$|A| \leq |B| \iff \exists f : A \to B \text{ such that } f \text{ is injective.}$$

We write $|A| \geq |B|$ if $|B| \leq |A|$. Furthermore,

$$|A| < |B| \iff |A| \leq |B| \text{ and } |A| \neq |B|,$$

and $|A| > |B|$ if $|B| < |A|$

Definition 1.3 (Equinumerous Sets). [def:same-cardinality] Sets A and B are *equinumerous*, written $A \sim B$, if and only if their cardinalities are equal:

$$A \sim B \iff |A| = |B|$$

LaTeX: A\equinum B

Theorem 1.1 (Bijection Characterization). [thm:bijection-cardinality]

$$|A| = |B| \iff \exists f : A \to B \textit{ such that } f \textit{ is bijective}$$

Lemma 1.1 (Finite Subsets of Equal Cardinality). [lem:finite-subset-equal-cardinality] *Let $A \subseteq B$, where B is finite. If $|A| = |B|$, then $A = B$*

Proof. If $A \neq B$, choose $b \in B \setminus A$. Then $A \subseteq B \setminus \{b\}$, so $|A| \leq |B| - 1 < |B|$, a contradiction $\square$

Lemma 1.2 (Mapping Characterization). [lem:cardinal-order-mappings] *Assume $B \neq \emptyset$ and the Axiom of Choice. Then*

$$\begin{aligned}
|A| \leq |B| &\iff \exists f : A \to B \textit{ injective},\\
|A| < |B| &\iff \exists f : A \to B \textit{ injective but not bijective},\\
|A| \geq |B| &\iff \exists f : A \to B \textit{ surjective},\\
|A| > |B| &\iff \exists f : A \to B \textit{ surjective but not bijective}
\end{aligned}$$

Definition 1.4 (Equivalence Relation). [def:equivalence-relation] A relation $\sim$ on a set A is called an *equivalence relation* if it satisfies the following properties:

1. Reflexive: $\forall a \in A, a \sim a$
2. Symmetric: $\forall a, b \in A, a \sim b \implies b \sim a$
3. Transitive: $\forall a, b, c \in A, a \sim b, b \sim c \implies a \sim c$

Theorem 1.2 (Cardinality is an Equivalence Relation). [thm:equivalence-relation-cardinality] *The Cardinality relation $\sim$ on the set of all sets is an equivalence relation*

Example 1.1. $\mathbb{N} \sim \mathbb{N}_1$. Witness: $f : \mathbb{N} \to \mathbb{N}_1, f(n) = n + 1$

Remark 1.1. For infinite sets, the cardinality maybe still the same even if we remove some (or a finite number of) elements from the set

Theorem 1.3 (Integers and Natural Numbers are Equinumerous). [thm:integers-natural-equinum] $\mathbb{Z} \sim \mathbb{N}$. *Witness:*

$$f = \begin{cases} n/2 & \textit{if } n \textit{ is even} \\ -(n+1)/2 & \textit{if } n \textit{ is odd} \end{cases}$$

2 Power Sets

Definition 2.1 (Power Set)**.** [def:power-set] The *power-set constructor* assigns to each set its set of all subsets

$$\mathcal{P} : \mathbb{T} \to \mathbb{T} \equiv A \mapsto \{B \mid B \subseteq A\}$$

`LaTeX: \PowerSet (A)`
`LaTeX: \Type`

Example 2.1. Let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$$

Theorem 2.1 (Cardinality of Finite Power Set)**.** [thm:cardinality-finite-power-set] *For any finite set A, $|\mathcal{P}(A)| = 2^{|A|}$*

Lemma 2.1 (Singleton in the Power Set)**.** [lem:singleton-in-power-set] *For every set A,*

$$\forall a \in A, \quad \{a\} \in \mathcal{P}(A)$$

Proof. If $a \in A$, then $\{a\} \subseteq A$. Hence $\{a\} \in \mathcal{P}(A)$ □

Theorem 2.2 (Cantor's Theorem)**.** [thm:cantor-power-set] *There does not exist a surjective function $f : A \to \mathcal{P}(A)$*

Proof. Assume for contradiction that there exists a surjective function $f : A \to \mathcal{P}(A)$. This means $\forall B \in \mathcal{P}(A), \exists a \in A$ such that $f(a) = B$
We can seperate all elements in A into two groups:

$$A_1 = \{a \in A : a \notin f(a)\}$$
$$A_2 = \{a \in A : a \in f(a)\}$$

Notice that $A_1 \subseteq A$, so $A_1 \in \mathcal{P}(A)$
And, since f is surjective, there exists $\alpha \in A$ such that $f(\alpha) = A_1$
According to PNP Axiom, α must be in either A_1 or A_2
If $\alpha \in A_1$, then by definition $\alpha \notin f(\alpha) = A_1$, contradicting $\alpha \in A_1$
If $\alpha \in A_2$, then by definition $\alpha \in f(\alpha) = A_1$. Hence $\alpha \in A_1$, whose definition gives $\alpha \notin f(\alpha)$, also a contradiction
Both possible cases lead to a contradiction. Therefore, the assumed surjective function $f : A \to \mathcal{P}(A)$ does not exist □

Corollary 2.1 (Power Set Strictly Larger than Original Set)**.** [cor:power-set-strictly-larger] *For any set A, $|A| < |\mathcal{P}(A)|$*

Remark 2.1 (Largest Cardinality does not Exist). [rem:largest-cardinality-does-not-exist] There is no largest cardinality for sets since for any set A, $|A| < |\mathcal{P}(A)|$

3 Images

Definition 3.1 (Image Operator)**.** [def:image] For sets A and B, the *image operator* is the mapping

$$(-)^{\text{img}} : (A \to B) \to \big(\mathcal{P}(A) \to \mathcal{P}(B)\big)$$

defined by

$$f \mapsto (C \mapsto \{f(x) : x \in C\})$$

`LaTeX: \img {f}(C)`

Corollary 3.1 (Image of a Subset). [cor:image-of-subset] *For a mapping $f : A \to B$ and a set $C \subseteq A$,*

$$f^{\text{img}}(C) = \{f(x) : x \in C\} \subseteq B$$

Lemma 3.1 (Monotonicity of Images). [lem:image-monotonicity] *Let $g : X \to Y$ and let $E, F \subseteq X$. Then*

$$E \subseteq F \implies g^{\text{img}}(E) \subseteq g^{\text{img}}(F)$$

Proof. If $y \in g^{\text{img}}(E)$, then $y = g(x)$ for some $x \in E$. Since $E \subseteq F$, we have $x \in F$, and therefore $y \in g^{\text{img}}(F)$ □

Definition 3.2 (Preimage Operator). [def:pre-image] For sets A and B, the *preimage operator* is the mapping

$$(-)^{\text{preimg}} : (A \to B) \to \big(\mathcal{P}(B) \to \mathcal{P}(A)\big)$$

defined by

$$f \mapsto (D \mapsto \{x \in A : f(x) \in D\})$$

`LaTeX: \preimg {f}(D)`

Corollary 3.2 (Preimage of a Subset). [cor:preimage-of-subset] *For a mapping $f : A \to B$ and a set $D \subseteq B$,*

$$f^{\text{preimg}}(D) = \{x \in A : f(x) \in D\} \subseteq A$$

Lemma 3.2 (Monotonicity of Pre-images). [lem:preimage-monotonicity] *Let $g : X \to Y$ and let $E, F \subseteq Y$. Then*

$$E \subseteq F \implies g^{\text{preimg}}(E) \subseteq g^{\text{preimg}}(F)$$

Remark 3.1 (Image of Pre-image). [rem:image-of-pre-image] In general cases, for a mapping $f : A \to B$,

$$C \subseteq A, \not\Rightarrow f^{\text{preimg}}(f^{\text{img}}(C)) = C$$
$$D \subseteq B, \not\Rightarrow f^{\text{img}}(f^{\text{preimg}}(D)) = D$$

Pf. Counterexamples. Define $f : \mathbb{N} \to \mathbb{B}$ by

$$f(n) = \begin{cases} \text{ff}, & n = 0, \\ \text{tt}, & n \in \mathbb{N}_1. \end{cases}$$

Then

$$f^{\text{preimg}}(f^{\text{img}}(\{1\})) = f^{\text{preimg}}(\{\text{tt}\}) = \mathbb{N}_1 \neq \{1\}$$

Define $g : \mathbb{B} \to \mathbb{N}$ by

$$g(\text{tt}) = 0, \qquad g(\text{ff}) = 1.$$

Since 2 is not in the range of g,

$$g^{\text{img}}(g^{\text{preimg}}(\{2\})) = g^{\text{img}}(\emptyset) = \emptyset \neq \{2\}$$

□

Remark 3.2 (What to do when a result is not true?). [rem:how-to-treat-false-results] When a result such as the equalities in remark 3.1 is not true, we may:

- strengthen the assumptions, as in proposition 3.2;
- narrow down the conclusion, as in proposition 3.1

Proposition 3.1 (Subset Relation for Pre-image of Image). [prop:preimage-image-subset] *In general cases, for a mapping $f : A \to B$,*

$$C \subseteq A, f^{\text{preimg}}(f^{\text{img}}(C)) \supseteq C$$
$$D \subseteq B, f^{\text{img}}(f^{\text{preimg}}(D)) \subseteq D$$

Proposition 3.2 (Pre-image of Image for Bijection Mapping). [prop:bijection-preimage-image] *If $f : A \to B$ is a bijection mapping, then*

$$C \subseteq A, \implies f^{\text{preimg}}(f^{\text{img}}(C)) = C$$
$$D \subseteq B, \implies f^{\text{img}}(f^{\text{preimg}}(D)) = D$$

4 Nested Sequence of Sets

Definition 4.1 (Nested Sequence of Sets). [def:nested-sequence-of-sets] Let X be a set and let $I = \mathbb{N}$ or $I = \mathbb{N}_1$. A sequence of subsets $A : I \to \mathcal{P}(X)$ is *nested* if

$$\forall n \in I, \quad A(n+1) \subseteq A(n)$$

Definition 4.2 (Set of Nested Sequences). [def:nested-sequences-set] For a set X and $I = \mathbb{N}$ or $I = \mathbb{N}_1$, the set of nested sequences of subsets of X is

$$\text{Nested}_I(X) \equiv \{A : I \to \mathcal{P}(X) \mid \forall n \in I,\ A(n+1) \subseteq A(n)\}$$

The default index set is $I = \mathbb{N}$

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LaTeX: \Nested (X)
LaTeX: \Nested [\NNOne ](X)
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Theorem 4.1 (Nested Sequence Containment). [thm:nested-sequence-containment] *Let $I = \mathbb{N}$ or $I = \mathbb{N}_1$, and let $A : \text{Nested}_I(X)$. Then*

$$\forall p, q \in I, \qquad p \leq q \implies A(q) \subseteq A(p)$$

Proof. Fix $p \in I$. We prove $A(p+r) \subseteq A(p)$ for $r \in \mathbb{N}$ by induction. The case $r = 0$ is immediate. If it holds for r, nesting gives

$$A(p+r+1) \subseteq A(p+r) \subseteq A(p)$$

For $q \geq p$, take $r = q - p \in \mathbb{N}$ to obtain the result □

Theorem 4.2 (Finite Nested Intersection Property). [thm:finite-nested-intersection] *Let $A : \text{Nested}_{\mathbb{N}_1}(X)$, and assume every $A(n)$ is finite and nonempty. Then*

$$\emptyset \neq \bigcap(A) \subseteq A(1)$$

Proof. According to the definition of definition indexed intersection, we want to show that

$$\exists x \in X, \forall n \in \mathbb{N}_1,\ x \in A(n)$$

Given that $\forall n \in \mathbb{N}_1,\ |A(n)| \in \mathbb{N}_1$ and $|A(1)| \geq |A(2)| \geq \cdots$, define

$$b : \mathbb{N}_1 \to \mathbb{N} \equiv n \mapsto |A(n)| - |A(n+1)|$$

For all $m, n \in \mathbb{N}_1$ with $m \leq n$,

$$\begin{aligned}\sum_{m}^{n}(b) &= (|A(m)| - |A(m+1)|) + \cdots + (|A(n)| - |A(n+1)|) \\ &= |A(m)| - |A(n+1)|\end{aligned}$$

Since $A(n+1) \neq \emptyset$, we have $|A(n+1)| \geq 1$. Taking $m = 1$,

$$\forall n \in \mathbb{N}_1, \quad \sum_{1}^{n}(b) = |A(1)| - |A(n+1)| \leq |A(1)| - 1$$

Since every $b(n)$ is a nonnegative integer, only finitely many $b(n)$ can be positive. Hence

$$\exists N_0 \in \mathbb{N}_1, \forall n \in \mathbb{N}_1, \, n \geq N_0 \implies b(n) = 0$$

For every $n \geq N_0$,

$$b(n) = 0 \implies |A(n)| = |A(n+1)|$$

Since $A(n+1) \subseteq A(n)$ and $A(n)$ is finite, finite subsets of equal cardinality gives $A(n+1) = A(n)$. Hence

$$\forall n \in \mathbb{N}_1, \quad n \geq N_0 \implies A(n) = A(N_0)$$

If $n < N_0$, then $A(N_0) \subseteq A(n)$ by nested sequence containment. Therefore $A(N_0) \subseteq A(n)$ for every $n \in \mathbb{N}_1$, and thus

$$A(N_0) \subseteq \bigcap(A)$$

The reverse inclusion holds because $A(N_0)$ is one of the sets in the intersection. Consequently,

$$\bigcap(A) = A(N_0)$$

which is finite and nonempty □

Remark 4.1 (Finiteness is Essential). [rem:finite-nested-intersection-finiteness] The conclusion can fail when the sets are infinite. For example, let

$$A : \mathrm{Nested}_{\mathbb{N}_1}(\mathbb{N}_1) \equiv n \mapsto \{m \in \mathbb{N}_1 \mid m \geq n\}$$

Every $A(n)$ is infinite and nonempty, and

$$A(1) \supseteq A(2) \supseteq A(3) \supseteq \cdots, \qquad \bigcap(A) = \emptyset$$

Proof. For each $n \in \mathbb{N}_1$, choose $x = n$. Then

$$\forall n \in \mathbb{N}_1, \quad \exists x \in \mathbb{N}_1 : \; x \in A(n)$$

For each $x \in \mathbb{N}_1$, choose $n = x + 1$. Then $x \notin A(n)$, so

$$\nexists x \in \mathbb{N}_1, \quad \forall n \in \mathbb{N}_1 : \; x \in A(n)$$

□

5 Intervals

Definition 5.1 (Interval)**.** [def:interval] Let X be a totally ordered set. A set $I \subseteq X$ is an *interval* if, whenever $x, y \in I$ with $x \leq y$, every $z \in X$ satisfying $x \leq z \leq y$ also belongs to I

Remark 5.1 (Intervals as Sets). [rem:intervals-as-sets] Every interval is a set contained in its ambient totally ordered set X; that is, every interval I satisfies $I \subseteq X$

Definition 5.2 (Open Interval)**.** [def:open-interval] Let X be a totally ordered set. For $a, b \in X$ with $a < b$, the *open interval* from a to b is

$$(a, b) = \{x \in X : a < x < b\}$$

Definition 5.3 (Closed Interval)**.** [def:closed-interval] Let X be a totally ordered set. For $a, b \in X$ with $a \leq b$, the *closed interval* from a to b is

$$[a, b] = \{x \in X : a \leq x \leq b\}$$

Lemma 5.1 (Endpoints of Contained Closed Intervals)**.** [lem:contained-interval-endpoints] *Let $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ be closed intervals in the same totally ordered set. If $I_1 \subseteq I_2$, then*

$$a_2 \leq a_1 \leq b_1 \leq b_2$$

Proof. Since $a_1, b_1 \in I_1 \subseteq I_2$, definition 5.3 gives $a_2 \leq a_1$ and $b_1 \leq b_2$
Since $I_1 = [a_1, b_1]$ is a closed interval, definition 5.3 requires $a_1 \leq b_1$
Combining these inequalities gives the result □

Lemma 5.2 (Endpoint Order of Nested Closed Intervals)**.** [lem:nested-interval-endpoint-order] *Let X be a totally ordered set, let $I : \mathrm{Nested}_{\mathbb{N}}(X)$, and let $a, b : \mathbb{N} \to X$ satisfy $I(k) = [a(k), b(k)]$ for every $k \in \mathbb{N}$. Then*

$$n \leq m \implies a(n) \leq a(m) \leq b(m) \leq b(n)$$

Proof. For $n \leq m$, nested sequence containment gives $I(m) \subseteq I(n)$. Applying endpoint order for contained closed intervals gives the inequalities □

Theorem 5.1 (Nested Intervals Property (NIP))**.** [thm:nip] *Let F be a complete ordered field, and let $I : \mathrm{Nested}_{\mathbb{N}}(F)$ with $I(n) = [a(n), b(n)]$, where $a, b : \mathbb{N} \to F$, be a nested sequence of closed intervals. Then*

$$\bigcap(I) \neq \emptyset$$

Proof. Let $A = \{a(n) : n \in \mathbb{N}\}$. For any $n, m \in \mathbb{N}$, either $n \leq m$ or $m \leq n$. By endpoint order for nested intervals, respectively,

$$a(n) \leq a(m) \leq b(m) \quad \text{or} \quad a(n) \leq b(n) \leq b(m)$$

Thus every $b(m)$ is an upper bound for A. Since $a(0) \in A$, the set A is nonempty and bounded above. By completeness, let $\alpha = \sup(A)$. The definition of supremum gives

$$\forall n \in \mathbb{N}, \quad a(n) \leq \alpha \leq b(n)$$

Therefore $\alpha \in I(n)$ for every $n \in \mathbb{N}$, so $\alpha \in \bigcap(I)$ by indexed intersection □

Corollary 5.1 (Nested Intervals Property in $\mathbb{R}$)**.** [cor:nip-real-numbers] *Let $I : \mathrm{Nested}_{\mathbb{N}}(\mathbb{R})$ with $I(n) = [a(n), b(n)]$, where $a, b : \mathbb{N} \to \mathbb{R}$. Then*

$$\bigcap(I) \neq \emptyset$$

Proof. By the Axiom of Completeness and the definition of a complete ordered field, $\mathbb{R}$ is complete. The conclusion follows from the Nested Intervals Property □

Remark 5.2 (Intervals can be Infinitely Small). [rem:inf-small-intervals] Let F be an ordered field and let $a, b \in F$ with $a \leq b$. If $\forall \varepsilon \in F_+,\ b - a < \varepsilon$, then $a = b$

6 Decimal Expansion

Theorem 6.1 (Decimal Expansion)**.** [thm:decimal-representation] *Let $D = \{0, 1, \dots, 9\}$. Every digit sequence $a : \mathbb{N}_1 \to D$ determines a unique $x \in [0, 1]$:*

$$\forall a : \mathbb{N}_1 \to D, \quad \exists! x \in [0, 1], \qquad x = \sum_{k=1}^{\infty} a_k 10^{-k}$$

Proof. We call the summation of the first n decimals s_n, ie

$$s_n = \sum_{k=1}^{n} a_k \, 10^{-k}$$

Since $\forall k \in \mathbb{N}_1, 0 \le a_k \le 9$, for all $n, m \in \mathbb{N}_1$ with $n \le m$,

$$\begin{aligned} s_m &= s_n + \sum_{k=n+1}^{m} a_k 10^{-k} \\ &\le s_n + 9 \sum_{k=n+1}^{m} 10^{-k} \\ &= s_n + 10^{-n} - 10^{-m} < s_n + 10^{-n} \end{aligned}$$

In other words,

$$\forall n, m \in \mathbb{N}_1, \quad n \le m \implies s_m \in [s_n, s_n + 10^{-n}]$$

By the same argument, we can also show that

$$\forall n, m \in \mathbb{N}_1, \quad n \le m \implies s_m + 10^{-m} \in [s_n, s_n + 10^{-n}]$$

Where we can construct the following nested intervals

$$I_n = [s_n, s_n + 10^{-n}], \qquad I_1 \supseteq I_2 \supseteq \cdots, \qquad x = \sum_{n=\mathbb{N}_1} a_n 10^{-k} = \bigcap_{n \in \mathbb{N}_1} I_n$$

According to the Nested Interval Theorem, after reindexing,

$$\bigcap_{n \in \mathbb{N}_1} I_n \neq \emptyset$$

Also, notice that $I_n = [s_n, s_n + 10^{-n}]$, $s_n + 10^{-n} - s_n = 10^{-n}$. According to the Remark, we can conclude:

$$\forall \varepsilon \in \mathbb{R}_+, \ \exists n \in \mathbb{N}_1 : \ I_n = [s_n, s_n + 10^{-n}] \text{ and } (s_n + 10^{-n}) - s_n = 10^{-n} < \varepsilon \implies \exists! x \in \bigcap_{n \in \mathbb{N}_1} I_n$$

Thus x is the unique number represented by the digit sequence a □

Remark 6.1 (Decimal Expansion is not Unique). [rem:decimal-representation-not-unique] The decimal expansion of a real number is not unique. For example, $0.8999\ldots = 0.9000\ldots$

Theorem 6.2 (Uniqueness of Canonical Decimal Expansions)**.** [thm:canonical-decimal-uniqueness] *Let $D = \{0, 1, \dots, 9\}$. For every $x \in [0, 1)$, there exists a unique digit sequence $a : \mathbb{N}_1 \to D$ which is not eventually equal to 9 and satisfies*

$$x = \sum_{n=1}^{\infty} a(n) 10^{-n}.$$

Here, "not eventually equal to 9" means

$$\neg \exists N \in \mathbb{N}_1 \ \forall n \ge N, \quad a(n) = 9$$

Proof. **TODO** □

7 Archimedean Property

Theorem 7.1 (Archimedean Property). [thm:archimedean-property] *$\mathbb{N}$ is not bounded above in $\mathbb{R}$, in other words, $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, n > x$*

Proof. Assume for the sake of contradiction that $\mathbb{N}$ is bounded above in $\mathbb{R}$
Since $0 \in \mathbb{N}$, we have $\mathbb{N} \neq \emptyset$. By the Axiom of Completeness, $\mathbb{N}$ has a supremum, denoted as $\alpha = \sup \mathbb{N}$
By the definition of epsilon criteria, we have:

$$\forall x \in \mathbb{N}, x \leq \alpha, \quad \forall \varepsilon \in \mathbb{R}_+, \exists x \in \mathbb{N}, \alpha - \varepsilon < x$$

Taking $\varepsilon = 1$, we obtain

$$\exists n \in \mathbb{N} : \quad \alpha - 1 < n \implies \alpha < n + 1$$

Since $n + 1 = S(n) \in \mathbb{N}$, this contradicts that α is an upper bound of $\mathbb{N}$, namely

$$\forall m \in \mathbb{N}, \quad m \leq \alpha.$$

Therefore, $\mathbb{N}$ is not bounded above in $\mathbb{R}$ □

Theorem 7.2 (Archimedean Property (Reciprocal Form)). [thm:archimedean-reciprocal-form]

$$\forall y \in \mathbb{R}_+, \exists n \in \mathbb{N}_1, 1/n < y$$

Proof. Take $x = 1/y$. Since $y > 0$, we have $x > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$ with $n > x$. Since $x > 0$, we have $n > 0$, hence $n \in \mathbb{N}_1$. Then $1/n < 1/x = y$ □

Definition 7.1 (Dense Subset). [def:dense-subset] Let $D \subseteq \mathbb{R}$. We say that D is *dense in* $\mathbb{R}$ if

$$\forall a, b \in \mathbb{R}, \quad a < b \implies \exists d \in D : \quad a < d < b$$

Theorem 7.3 (Density of the Rational Numbers). [thm:rationals-dense-in-reals] *$\mathbb{Q}$ is dense in $\mathbb{R}$*

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. Then $b - a > 0$, so by the Archimedean property (reciprocal form) there exists $n \in \mathbb{N}_1$ with

$$\frac{1}{n} < b - a \iff 1 < n(b - a) \iff 1 < nb - na$$

Let $m = \lfloor na \rfloor$, so that $m \leq na < m + 1$
Set $r \equiv (m + 1)/n \in \mathbb{Q}$
Notice that

$$na < m + 1 \implies \frac{na}{n} < \frac{m + 1}{n} \implies a < r$$

$$1 < nb - na \implies 1 + na < nb \implies 1 + m \leq 1 + na < nb \implies \frac{m + 1}{n} < b \implies r < b$$

Therefore $r \in \mathbb{Q} \cap (a, b)$
So, $\forall a, b \in \mathbb{R}, a < b, \exists r \in \mathbb{Q} : a < r < b$ □

Theorem 7.4 (Density of the Irrational Numbers). [thm:irrationals-dense-in-reals] *$\mathbb{R} \setminus \mathbb{Q}$ (ie, $\mathbb{I}$) is dense in $\mathbb{R}$*

Proof. **TODO** □

Remark 7.1. $\mathbb{Q}$ and $\mathbb{I}$ are both dense in $\mathbb{R}$. It does not mean that $\mathbb{Q}$ and $\mathbb{I}$ are having the same size. The size of infinite sets are defined in

8 Countability

Definition 8.1 (Countable Infinite Set)**.** [def:countable-infinite-set] A set A is called *countably infinite* if $N \sim A$

Definition 8.2 (Countable Set)**.** [def:countable-set] A set A is called *countable* if it is either finite or countably infinite

Theorem 8.1 (Countability of the Rationals)**.** [thm:rationals-natural-equinum] $\mathbb{Q} \sim \mathbb{N}$

Proof. List the positive rationals in the following pattern, starting from $1/1$ and replacing each a/b by $a/(a+b)$ followed by $(a+b)/b$ to form the next row:

$$\frac{1}{1}; \qquad \frac{1}{2}, \frac{2}{1}; \qquad \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}; \qquad \dots$$

Read row by row; the rows contain $1, 2, 4, \dots$ elements. Let $r_1, r_2, \dots$ be this enumeration. Since every positive rational occurs exactly once, a witness $f : \mathbb{N} \to \mathbb{Q}$ is

$$f(n) = \begin{cases} 0, & n = 0, \\ r_{(n+1)/2}, & n \text{ is odd}, \\ -r_{n/2}, & n \text{ is even and } n > 0 \end{cases}$$

□

Theorem 8.2 (Equinumerosity of Intervals)**.** [thm:interval-equinum] *Let F be an ordered field, and let $a, b, c, d \in F$ satisfy $a < b$ and $c < d$. Then the open intervals (a, b) and (c, d) are equinumerous*

Proof. Witness:

$$f : (a, b) \to (c, d), f(x) = \frac{d-c}{b-a}(x - a) + c$$

□

Theorem 8.3 (Intervals and the Real Line)**.** [thm:interval-real-line-equinum] *Every interval which is not a single point is equinumerous to $\mathbb{R}$*

Proof. For $(-\pi/2, \pi/2)$, the function $f(x) = \tan(x)$ is a bijection from $(-\pi/2, \pi/2)$ to $\mathbb{R}$
Notice that for every interval (a, b) which is not a single point, we can find a bijection from (a, b) to $(-\pi/2, \pi/2)$ by Theorem 8.2. Since bijections are transitive, we can conclude that every interval (a, b) which is not a single point is equinumerous to $\mathbb{R}$ □

Theorem 8.4 ($\mathbb{Z}, \mathbb{Q}$ are Countable)**.** [thm:integers-rationals-countable] *$\mathbb{Z}$ and $\mathbb{Q}$ are countable since they are equinumerous to $\mathbb{N}$ according to Theorem 1.3 and Theorem 8.1*

Theorem 8.5 (Cantor's Theorem)**.** [thm:cantor-theorem] *The interval $(0, 1)$ is uncountable*

Proof. Since $\mathbb{N} \sim \mathbb{N}_1$, suppose there exists a bijection $f : \mathbb{N}_1 \to (0, 1)$. By Theorem 6.2, choose the unique decimal expansion of each $f(n)$ which is not eventually equal to 9. This defines a function $g : \mathbb{N}_1 \to \mathbb{N}_1 \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Then, we can define another function of decimal expansion

$$h(n) = \Upsilon(g(n)(n)), \qquad n \in \mathbb{N}_1 \tag{1}$$

$$\Upsilon = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 1), (9, 1)\} \tag{2}$$

Then, we can construct

$$x = \sum_{n=1}^{\infty} h(n) 10^{-n}.$$

Since $h(n) \in \{1, 2, \dots, 8\}$, we have $x \in (0, 1)$, and this is its unique canonical decimal expansion by Theorem 6.2. For every $n \in \mathbb{N}_1$, $h(n) \neq g(n)(n)$, so $x \neq f(n)$. Hence x is not in the image of f, contradicting the assumption that f is a bijection. Therefore, $(0, 1)$ is uncountable □

Theorem 8.6 ($\mathbb{R}$ is Uncountable)**.** [thm:reals-uncountable] *$\mathbb{R}$ is uncountable*

Proof. Assume that $\mathbb{R}$ is countable. According to Theorem 8.3, we can conclude that $(0,1) \sim \mathbb{R} \sim \mathbb{N}$. According to Cantor's Theorem, $(0,1)$ is uncountable, which contradicts the assumption that $\mathbb{R}$ is countable. Therefore, $\mathbb{R}$ is uncountable □

Theorem 8.7 (Countable Union of Countable Sets)**.** [thm:countable-union-of-countable-sets] *Given n countable sets $A_1, A_2, \ldots, A_n$, the union $\bigcup_{i=1}^{n} A_i$ is countable*

Proof. We can prove this theorem by induction
Let A and B be two countable sets. If A and B are finite, trivially it is countable □