Chapter 8: WKB Approximation

Quantum Mechanics

1 Tunneling: Intuition and Terminology

Remark 1.1 (Physical Picture of Tunneling). [rem:tunneling-intuition] **TODO**

Convention 1.1 (Classically Allowed Region). [conv:allowed-region] For fixed E and V, the positions x satisfying $E > V(x)$ form the *classically allowed region*

Convention 1.2 (Classically Forbidden Region). [conv:forbidden-region] For fixed E and V, the positions x satisfying $E < V(x)$ form the *classically forbidden region*

Definition 1.1 (Turning Point). [def:turning-point] For fixed E and V, a *turning point* is a position x_1 satisfying

$$V(x_1) = E$$

Definition 1.2 (Simple Turning Point). [def:simple-turning-point] A turning point x_1 is *simple* if

$$\mathrm{D}(V)(x_1) \neq 0$$

Definition 1.3 (Transmission Coefficient). [def:transmission-coefficient] **TODO**

2 WKB Wavefunctions in the Allowed Region

Definition 2.1 (Classical Momentum Function). [def:classical-momentum] **TODO**

Hypothesis 2.1 (Slowly Varying Amplitude). [hypo:slowly-varying-amplitude] ***TODO***

Theorem 2.1 (Allowed-Region WKB Form). [thm:wkb-allowed-region] *On an allowed interval I, let $p(x) = \sqrt{2m \cdot (E - V(x))}$ and fix $x_0 \in I$; for slowly varying amplitude, the WKB branches are*

$$\psi_{\pm} : \mathbb{R} \to (I \to \mathbb{C}), \qquad A \mapsto \left(x \mapsto \frac{A}{\sqrt{p(x)}} \cdot \exp\left(\pm \frac{i}{\hbar} \cdot \left[\mathrm{D}^{-1}(p) \right]_{x_0}^{x} \right) \right)$$

Proof. **TODO** □

Remark 2.1 (Validity of the WKB Approximation). [rem:wkb-validity] **TODO**

Theorem 2.2 (Quantization Between Two Infinite Walls). [thm:hard-wall-quantization] ***TODO***

Proof. **TODO** □

3 WKB Wavefunctions in the Forbidden Region

Definition 3.1 (Forbidden-Region Momentum Magnitude). [def:forbidden-momentum] **TODO**

Theorem 3.1 (Forbidden-Region WKB Form). [thm:wkb-forbidden-region] *On a forbidden interval $[0, a]$, let $q(x) = \sqrt{2m \cdot (V(x) - E)}$; neglecting the growing term, the WKB family is*

$$\psi_{\mathrm{dec}} : \mathbb{R} \to ([0, a] \to \mathbb{R}), \qquad D \mapsto \left(x \mapsto \frac{D}{\sqrt{q(x)}} \cdot \exp\left(-\frac{1}{\hbar} \cdot \left[\mathrm{D}^{-1}(q) \right]_{0}^{x} \right) \right)$$

Proof. **TODO** □

Definition 3.2 (Barrier Action). [def:barrier-action] **TODO**

Theorem 3.2 (Leading WKB Tunneling Probability). [thm:wkb-transmission] ***TODO***

Proof. **TODO** □

Remark 3.1 (Abrupt Boundaries and Transmission Prefactors). [rem:transmission-prefactors] **TODO**

4 Gamow's Theory of Alpha Decay

Remark 4.1 (Alpha-Decay Barrier Model). [rem:alpha-decay-barrier] **TODO**

Theorem 4.1 (Barrier Penetration in Alpha Decay). [thm:alpha-decay-penetration] ***TODO***

Proof. **TODO** □

Remark 4.2 (Escape Attempts and Decay Rate). [rem:alpha-decay-attempts] **TODO**

Theorem 4.2 (Gamow's Estimate of the Decay Lifetime). [thm:gamow-lifetime] ***TODO***

Proof. **TODO** □

5 Airy Equation

The endpoint ∞ denotes an improper integral, real-analysis

Definition 5.1 (Airy Function Ai). [def:airy-ai]

$$\mathrm{Ai} : \mathbb{R} \to \mathbb{R} \equiv z \mapsto \frac{1}{\pi} \cdot \left[\mathrm{D}^{-1} \left(s \mapsto \cos\left(\frac{s^3}{3} + s \cdot z \right) \right) \right]_0^\infty$$

Definition 5.2 (Airy Function Bi). [def:airy-bi]

$$\mathrm{Bi} : \mathbb{R} \to \mathbb{R} \equiv z \mapsto \frac{1}{\pi} \cdot \left[\mathrm{D}^{-1} \left(s \mapsto \exp\left(-\frac{s^3}{3} + s \cdot z \right) + \sin\left(\frac{s^3}{3} + s \cdot z \right) \right) \right]_0^\infty$$

Theorem 5.1 (Airy Equation and Solutions). [thm:airy-equation-solution] *For twice-differentiable* $y : \mathbb{R} \to \mathbb{C}$*, the* Airy equation

$$\mathrm{D}^2(y)(z) = z \cdot y(z) \qquad (z \in \mathbb{R})$$

has general solution

$$y : \mathbb{R} \to \mathbb{C} \equiv z \mapsto a \cdot \mathrm{Ai}(z) + b \cdot \mathrm{Bi}(z), \qquad a, b \in \mathbb{C}$$

Proof. **TODO** □

Theorem 5.2 (Airy Asymptotic Forms). [thm:airy-asymptotics] *For* $z \gg 0$*,*

$$\mathrm{Ai}(z) \approx \frac{1}{2\sqrt{\pi} \cdot z^{1/4}} \cdot \exp\left(-\frac{2}{3} \cdot z^{3/2} \right),$$
$$\mathrm{Bi}(z) \approx \frac{1}{\sqrt{\pi} \cdot z^{1/4}} \cdot \exp\left(\frac{2}{3} \cdot z^{3/2} \right)$$

For $z \ll 0$*,*

$$\mathrm{Ai}(z) \approx \frac{1}{\sqrt{\pi} \cdot (-z)^{1/4}} \cdot \sin\left(\frac{2}{3} \cdot (-z)^{3/2} + \frac{\pi}{4} \right),$$
$$\mathrm{Bi}(z) \approx \frac{1}{\sqrt{\pi} \cdot (-z)^{1/4}} \cdot \cos\left(\frac{2}{3} \cdot (-z)^{3/2} + \frac{\pi}{4} \right)$$

Proof. **TODO** □

6 Turning Points and Connection Formulas

Remark 6.1 (Airy Functions and Overlap Matching). [rem:airy-matching] **TODO**

Theorem 6.1 (Upward-Sloping Connection Formula). [thm:upward-connection-formula] ***TODO***

Proof. **TODO** □

Theorem 6.2 (Downward-Sloping Connection Formula). [thm:downward-connection-formula] *Let V be smooth near x_1, with*

$$V(x_1) = E, \qquad \mathrm{D}(V)(x_1) < 0$$

Sufficiently close to x_1, the forbidden and allowed regions are

$$\begin{aligned} x < x_1 : &\quad V(x) > E \quad \textit{(forbidden)}, \\ x > x_1 : &\quad V(x) < E \quad \textit{(allowed)} \end{aligned}$$

Write $q(x) = \sqrt{2m \cdot (V(x) - E)}$ on the forbidden side and $p(x) = \sqrt{2m \cdot (E - V(x))}$ on the allowed side. On an interval I about x_1 with these regions, the connected WKB approximation is, for $D \in \mathbb{C}$,

$$\begin{aligned} &\psi' : I \setminus \{x_1\} \to \mathbb{C} \\ &\equiv x \mapsto \begin{cases} \dfrac{D}{\sqrt{q(x)}} \cdot \exp\left(-\dfrac{1}{\hbar} \cdot \left[\mathrm{D}^{-1}(q)\right]_x^{x_1}\right), & x < x_1, \\ \dfrac{2D}{\sqrt{p(x)}} \cdot \sin\left(\dfrac{1}{\hbar} \cdot \left[\mathrm{D}^{-1}(p)\right]_{x_1}^{x} + \dfrac{\pi}{4}\right), & x > x_1 \end{cases} \end{aligned}$$

The prime labels the approximation; these WKB forms apply away from the turning point

Proof. Choose $\delta > 0$ so the forbidden and allowed intervals are

$$I_F \equiv [x_1 - \delta, x_1), \qquad I_A \equiv (x_1, x_1 + \delta]$$

Write $q(x) = \sqrt{2m \cdot (V(x) - E)}$ on I_F and $p(x) = \sqrt{2m \cdot (E - V(x))}$ on I_A, with $B, C, D \in \mathbb{C}$
Choose the WKB branch that decays towards the left in the forbidden region:

$$\psi_F : I_F \to \mathbb{C} \equiv x \mapsto \frac{D}{\sqrt{q(x)}} \cdot \exp\left(-\frac{1}{\hbar} \cdot \left[\mathrm{D}^{-1}(q)\right]_x^{x_1}\right)$$

In the allowed region, use both WKB branches:

$$\begin{aligned} \psi_A : I_A \to \mathbb{C} \equiv x \mapsto \frac{1}{\sqrt{p(x)}} \cdot \Bigg(& B \cdot \exp\left(\frac{i}{\hbar} \cdot \left[\mathrm{D}^{-1}(p)\right]_{x_1}^{x}\right) \\ & + C \cdot \exp\left(-\frac{i}{\hbar} \cdot \left[\mathrm{D}^{-1}(p)\right]_{x_1}^{x}\right)\Bigg) \end{aligned}$$

The factors $1/\sqrt{q(x)}$ and $1/\sqrt{p(x)}$ diverge as x approaches x_1 from their sides, so these WKB forms fail near the turning point; a local matching solution is needed
Seek a local patch

$$\psi_P : [x_1 - \delta, x_1 + \delta] \to \mathbb{C}$$

Choose $x_1 \pm \delta$ in the overlap regions; to match the wavefunction and its one-sided first derivatives, require

$$\begin{cases} \psi_P(x_1 - \delta) \approx \psi_F(x_1 - \delta) \\ \mathrm{D}(\psi_P)(x_1 - \delta) \approx \mathrm{D}(\psi_F)(x_1 - \delta) \\ \psi_P(x_1 + \delta) \approx \psi_A(x_1 + \delta) \\ \mathrm{D}(\psi_P)(x_1 + \delta) \approx \mathrm{D}(\psi_A)(x_1 + \delta) \end{cases}$$

According to TISE, for $x \in (x_1 - \delta, x_1 + \delta)$,

$$-\frac{\hbar^2}{2m} \cdot \mathrm{D}^2(\psi_P)(x) + V(x) \cdot \psi_P(x) = E \cdot \psi_P(x)$$

First-order Taylor expansion about the turning point gives

$$V(x) \approx V(x_1) + \mathrm{D}(V)(x_1) \cdot (x - x_1) = E + \mathrm{D}(V)(x_1) \cdot (x - x_1)$$

Substitute into TISE and cancel $E \cdot \psi_P(x)$:

$$-\frac{\hbar^2}{2m} \cdot \mathrm{D}^2(\psi_P)(x) + \big(E + \mathrm{D}(V)(x_1) \cdot (x - x_1)\big) \cdot \psi_P(x) \approx E \cdot \psi_P(x)$$

$$-\frac{\hbar^2}{2m} \cdot \mathrm{D}^2(\psi_P)(x) + \mathrm{D}(V)(x_1) \cdot (x - x_1) \cdot \psi_P(x) \approx 0$$

Since $\mathrm{D}(V)(x_1) < 0$, define

$$\alpha \equiv \left(-\frac{2m}{\hbar^2} \cdot \mathrm{D}(V)(x_1)\right)^{1/3} > 0$$

Then

$$\mathrm{D}^2(\psi_P)(x) + \alpha^3 \cdot (x - x_1) \cdot \psi_P(x) \approx 0$$

$$\mathrm{D}^2(\psi_P)(x) \approx -\alpha^3 \cdot (x - x_1) \cdot \psi_P(x)$$

Set $z = -\alpha \cdot (x - x_1)$ and write $\psi_P(x) = y(z)$; the chain rule gives

$$\mathrm{D}^2(\psi_P)(x) = \alpha^2 \cdot \mathrm{D}^2(y)(z)$$

$$\alpha^2 \cdot \mathrm{D}^2(y)(z) \approx \alpha^2 \cdot z \cdot y(z) \quad \Longrightarrow \quad \mathrm{D}^2(y)(z) \approx z \cdot y(z)$$

By the Airy general solution, the local approximation is

$$\psi_P(x) \approx a \cdot \mathrm{Ai}\big(-\alpha \cdot (x - x_1)\big) + b \cdot \mathrm{Bi}\big(-\alpha \cdot (x - x_1)\big), \qquad a, b \in \mathbb{C}$$

Use the Airy asymptotic forms in overlap regions where $|z| \gg 1$ while the linear approximation to V remains valid
On the forbidden side, $x < x_1$ and $z \gg 0$:

$$q(x) \approx \hbar \cdot \alpha \cdot \sqrt{z}, \qquad \frac{1}{\hbar} \cdot \left[\mathrm{D}^{-1}(q)\right]_x^{x_1} \approx \frac{2}{3} \cdot z^{3/2}$$

$$\psi_P(x) \approx \frac{1}{z^{1/4}} \cdot \left[\frac{a}{2\sqrt{\pi}} \cdot e^{-\frac{2}{3}z^{3/2}} + \frac{b}{\sqrt{\pi}} \cdot e^{\frac{2}{3}z^{3/2}}\right],$$

$$\psi_F(x) \approx \frac{1}{z^{1/4}} \cdot \frac{D}{\sqrt{\hbar \cdot \alpha}} \cdot e^{-\frac{2}{3}z^{3/2}}$$

Matching the decaying and growing coefficients gives

$$\begin{cases} \dfrac{a}{2\sqrt{\pi}} = \dfrac{D}{\sqrt{\hbar \cdot \alpha}} \\ \dfrac{b}{\sqrt{\pi}} = 0 \end{cases}$$

Hence

$$a = \frac{2\sqrt{\pi} \cdot D}{\sqrt{\hbar \cdot \alpha}}, \qquad b = 0$$

On the allowed side, $x > x_1$ and $z \ll 0$:

$$p(x) \approx \hbar \cdot \alpha \cdot \sqrt{-z}, \qquad \frac{1}{\hbar} \cdot \left[\mathrm{D}^{-1}(p)\right]_{x_1}^{x} \approx \frac{2}{3} \cdot (-z)^{3/2}$$

$$\psi_P(x) \approx \frac{1}{\sqrt{\pi} \cdot (-z)^{1/4}} \cdot \left[a \cdot \sin\left(\frac{2}{3} \cdot (-z)^{3/2} + \frac{\pi}{4}\right) + b \cdot \cos\left(\frac{2}{3} \cdot (-z)^{3/2} + \frac{\pi}{4}\right) \right],$$

$$\psi_A(x) \approx \frac{1}{\sqrt{\hbar \cdot \alpha} \cdot (-z)^{1/4}} \cdot \left[B \cdot e^{i\frac{2}{3}(-z)^{3/2}} + C \cdot e^{-i\frac{2}{3}(-z)^{3/2}} \right]$$

Using

$$a \cdot \sin\theta + b \cdot \cos\theta = \frac{b - ia}{2} \cdot e^{i\theta} + \frac{b + ia}{2} \cdot e^{-i\theta}$$

and matching the two oscillatory coefficients gives

$$\begin{cases} \dfrac{B}{\sqrt{\hbar \cdot \alpha}} = \dfrac{b - ia}{2\sqrt{\pi}} \cdot e^{i\pi/4} \\ \dfrac{C}{\sqrt{\hbar \cdot \alpha}} = \dfrac{b + ia}{2\sqrt{\pi}} \cdot e^{-i\pi/4} \end{cases}$$

Substituting a, b gives

$$\begin{aligned} B &= \sqrt{\hbar \cdot \alpha} \cdot \frac{-i}{2\sqrt{\pi}} \cdot \frac{2\sqrt{\pi} \cdot D}{\sqrt{\hbar \cdot \alpha}} \cdot e^{i\pi/4} \\ &= -i \cdot D \cdot e^{i\pi/4} = D \cdot e^{-i\pi/4}, \end{aligned}$$

$$\begin{aligned} C &= \sqrt{\hbar \cdot \alpha} \cdot \frac{i}{2\sqrt{\pi}} \cdot \frac{2\sqrt{\pi} \cdot D}{\sqrt{\hbar \cdot \alpha}} \cdot e^{-i\pi/4} \\ &= i \cdot D \cdot e^{-i\pi/4} = D \cdot e^{i\pi/4} \end{aligned}$$

Substituting back into the local patch gives

$$\psi_P \approx \left(x \mapsto \frac{2\sqrt{\pi} \cdot D}{\sqrt{\hbar \cdot \alpha}} \cdot \mathrm{Ai}\big(-\alpha \cdot (x - x_1)\big) \right)$$

Substituting B, C into the allowed-region form gives

$$\begin{aligned} \psi_A(x) &\approx \frac{D}{\sqrt{p(x)}} \cdot \left[e^{-i\pi/4} \cdot \exp\left(\frac{i}{\hbar} \cdot \left[\mathrm{D}^{-1}(p)\right]_{x_1}^{x}\right) + e^{i\pi/4} \cdot \exp\left(-\frac{i}{\hbar} \cdot \left[\mathrm{D}^{-1}(p)\right]_{x_1}^{x}\right) \right] \\ &= \frac{2D}{\sqrt{p(x)}} \cdot \cos\left(\frac{1}{\hbar} \cdot \left[\mathrm{D}^{-1}(p)\right]_{x_1}^{x} - \frac{\pi}{4}\right) \\ &= \frac{2D}{\sqrt{p(x)}} \cdot \sin\left(\frac{1}{\hbar} \cdot \left[\mathrm{D}^{-1}(p)\right]_{x_1}^{x} + \frac{\pi}{4}\right) \end{aligned}$$

Combining ψ_F and ψ_A gives the stated ψ' □
