Chapter 6: Symmetries

Quantum Mechanics

1 Introduction to Symmetry

Definition 1.1 (Symmetry)**.** [def:symmetry] A *symmetry* of a system is some transformation, such that the system is invariant before and after the transformation.

Definition 1.2 (Symmetry of Quantum System)**.** [def:symmetry-qm] A *symmetry* of a quantum system is a transformation that leaves the Hamiltonian invariant.

Definition 1.3 (Types of Symmetry)**.** [def:symmetry-types] There are several types of symmetries in quantum mechanics, including:

- *Translation symmetry*: The system is invariant under some spatial translations.
- *Rotation symmetry*: The system is invariant under some rotations.
- *Reflection symmetry*: The system is invariant under reflections on some axis/planes.
- *Inversion symmetry*: The system is invariant under inversion of all coordinates.
- *Time-translation symmetry*: The system is invariant under time shifts.

Theorem 1.1 (Symmetry Condition on the Hamiltonian)**.** [thm:symmetry-condition-hamiltonian] *$\hat{U}$ is a definition 1.2 of $\hat{H}$ iff*

$$\hat{U}^\dagger \hat{H} \hat{U} = \hat{H}.$$

Corollary 1.1 (Commutator Form)**.** [cor:commutator-form-symmetry] *For unitary $\hat{U}$,*

$$\hat{U}^\dagger \hat{H} \hat{U} = \hat{H} \iff \operatorname{comm}\Big(\hat{U},\, \hat{H}\Big) = 0.$$

Proof. Left-multiply by $\hat{U}$ and use $\hat{U}\hat{U}^\dagger = \hat{I}$:

$$\hat{U}^\dagger \hat{H} \hat{U} = \hat{H} \iff \hat{H}\hat{U} = \hat{U}\hat{H} \iff \operatorname{comm}\Big(\hat{U},\, \hat{H}\Big) = 0.$$

□

Remark 1.1 ("$\hat{A}$ Obtained After Applying $\hat{U}$")*.* [rem:conjugation-vocabulary] The phrase "$\hat{A}$ obtained after applying $\hat{U}$" refers to the conjugation $\hat{U}^\dagger \hat{A} \hat{U}$, since this preserves the expectation value.

2 Groups

Definition 2.1 (Group)**.** [def:group] A *group* consists of the following data and assumptions:

- A set G.
- A binary operation $\cdot : G \times G \to G$ (the codomain being G bakes in closure).
- An identity element $e \in G$.
- An inverse function $(\cdot)^{-1} : G \to G$, written $g \mapsto g^{-1}$.
- *Associativity*: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$.
- *Identity axiom*: $e \cdot g = g \cdot e = g$ for all $g \in G$.
- *Inverse axiom*: $g \cdot g^{-1} = g^{-1} \cdot g = e$ for all $g \in G$.

A collection with a well-behaved multiplication and inverses.

Example 2.1 (Real Numbers Under Addition)**.** [eg:reals-group] Choose:

- Set $G := \mathbb{R}$ (the real numbers).
- Operation $\cdot := +$ (ordinary addition).
- Identity $e := 0$.

- Inverse $a^{-1} := -a$.

Associativity, identity, and inverse axioms follow from the usual properties of addition.

Example 2.2 (Boolean Set Under XOR)**.** [eg:boolean-group] Choose:

- Set $G := \{\text{tt}, \text{ff}\}$ (equivalently $\{0, 1\}$).
- Operation $\cdot := \oplus$ (XOR, equivalent to addition mod 2).
- Identity $e := \text{ff}$ (equivalently 0).
- Inverse $g^{-1} := g$ (each element is self-inverse).

All axioms hold by direct verification.

Definition 2.2 (Continous Group)**.** [def:continuous-group] A group is *continuous* (a *Lie group*) if its elements are parametrized by a continuous parameter space, and multiplication and inversion vary smoothly with the parameters. The real numbers under addition is an example.

Definition 2.3 (Discrete Group)**.** [def:discrete-group] A group is *discrete* if its elements form a discrete set (finite or countable), with no continuous parameter connecting distinct elements. The Boolean set under XOR is an example.

Definition 2.4 (Continuous Symmetry)**.** [def:continuous-symmetry] A symmetry is *continuous* if its operators form a continuous group.

Definition 2.5 (Discrete Symmetry)**.** [def:discrete-symmetry] A symmetry is *discrete* if its operators form a discrete group.

3 Translation Operator

Definition 3.1 (Translation Operator)**.** [def:translation-operator] Intuitively, a *translation operator* $\hat{T}(\vec{r}_0)$ (translates a state by a vector $\vec{r}_0$) acts on a state $|\psi\rangle$ should let $|\psi\rangle$ be translated by a vector $\vec{r}$, i.e.

$$\hat{T}(\vec{r}_0)\,|\psi\rangle = |\psi(\vec{r} - \vec{r}_0)\rangle$$

Definition 3.2 (Translation Operator (1D))**.** [def:translation-operator-1d] In one dimension, the translation operator $\hat{T}(a)$ shifts a state by a real distance $a \in \mathbb{R}$. Its action on a position-space wavefunction is

$$\hat{T}(a)\,\psi(x) = \psi(x - a).$$

Theorem 3.1 (Translation Operators Form a Symmetry Group)**.** [thm:translation-symmetry-group] *Suppose $\hat{T}(a)$ is a symmetry operator of a Hamiltonian $\hat{H}$ for every $a \in \mathbb{R}$. Then the collection of translation operators forms a group under operator composition:*

- *Set $G := \{\hat{T}(a) : a \in \mathbb{R}\}$.*
- *Operation $\cdot$: operator composition, $\hat{T}(a) \cdot \hat{T}(b) = \hat{T}(a + b)$.*
- *Identity $e := \hat{T}(0) = \hat{I}$.*
- *Inverse $\hat{T}(a)^{-1} := \hat{T}(-a)$.*

All axioms follow from addition on $\mathbb{R}$.

Proof. □

Theorem 3.2 (Exponential Form of the Translation Operator)**.** [thm:translation-exponential] *The one-dimensional translation operator can be written as:*

$$\hat{T}(a) = e^{-ia\hat{p}/\hbar}$$

Proof. Expand the exponential in a Taylor series at 0

$$\hat{T}(a) = e^{-ia\hat{p}/\hbar} = \sum_{n=0}^{\infty} \frac{(-ia/\hbar)^n}{n!} \cdot \hat{p}^n = \sum_{n=0}^{\infty} \frac{(-ia\hat{p})^n}{n!\hbar^n}$$

According to the definition of the momentum operator, $\hat{p} = -i\hbar\frac{d}{dx}$, so we can substitute this into the series expansion:

$$\hat{T}(a) = \sum_{n=0}^{\infty} \frac{(-ia(-i\hbar\frac{d}{dx}))^n}{n!\hbar^n} = \sum_{n=0}^{\infty} \frac{(-a(\frac{d}{dx}))^n}{n!}$$

So,

$$\hat{T}(a)\psi(x) = \sum_{n=0}^{\infty} \frac{(-a(\frac{d}{dx}))^n}{n!}\psi(x) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}\psi^{(n)}(x)$$

Then, we do a Taylor expansion of $\psi(x)$ centered at $x + a$:

$$\psi(x) = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(x+a)}{n!}(x - (x+a))^n = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(x+a)}{n!}(-a)^n = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}\psi^{(n)}(x+a)$$

Substituting $x \to x - a$ in the previous identity,

$$\psi(x-a) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}\psi^{(n)}((x-a)+a) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}\psi^{(n)}(x).$$

Which means,

$$\hat{T}(a)\psi(x) = \psi(x-a)$$

□

Theorem 3.3 ($\hat{T}(a)$ is a Unitary Operator)**.** [thm:translation-unitary] *The translation operator $\hat{T}(a)$ is a unitary operator, i.e. $\hat{T}^\dagger(a) = \hat{T}^{-1}(a) = \hat{T}(-a)$.*

Remark 3.1 (Intuition about). [rem:intuition-about]

Proof. **TODO**: Proof to be written after clarifying preferred approach with the lecturer, direct from the action definition (definition 3.2), or via the exponential form (theorem 3.2). Email sent 2026-10-11. □

Theorem 3.4 (Position Operator After Applying Translation)**.** [thm:position-under-translation] *Applying $\hat{T}(a)$ to the position operator $\hat{x}$ (see remark 1.1) gives*

$$\hat{T}(a)^\dagger\, \hat{x}\, \hat{T}(a) \;=\; \hat{x} + a\hat{I}.$$

Proof. Define $\hat{x}' := \hat{T}(a)^\dagger\, \hat{x}\, \hat{T}(a)$, and consider its action on a wavefunction $\psi(x)$:

$$\hat{x}'\psi(x) = \hat{T}(a)^\dagger\, \hat{x}\, \hat{T}(a)\, \psi(x)$$

Since operators are associative, we have

$$\hat{x}'\psi(x) = \hat{T}(a)^\dagger\, \hat{x}\, (\hat{T}(a)\, \psi(x)) = \hat{T}(a)^\dagger\, (\hat{x}\, \psi(x-a)) = \hat{T}(a)^\dagger\, (x\, \psi(x-a))$$

Treat $g(x) = \hat{x}\, \psi(x-a)$, so, $T(a)\, g(x) = g(x) = x \hat{-} a\, \psi((x-a)-a)$.
Notice that $T(a)$ is a unitary operator, ie $T(a)^\dagger = T(a)^{-1} = T(-a)$, we have

$$\hat{x}'\psi(x) = \hat{T}(-a)\, (x\, \psi(x-a)) = (x+a)\, \psi((x+a)-a) = (x+a)\, \psi(x)$$

Rewrite the right-hand side in operator form:

$$(x+a)\,\psi(x) = x\,\psi(x) + a\,\psi(x) = \hat{x}\,\psi(x) + a\hat{I}\,\psi(x) = (\hat{x} + a\hat{I})\,\psi(x).$$

So $\hat{x}'\,\psi = (\hat{x} + a\hat{I})\,\psi$ for every ψ. Since two operators agreeing on all inputs are equal, we conclude

$$\hat{x}' = \hat{x} + a\hat{I}.$$

□

Theorem 3.5 (Momentum Operator After Applying Translation)**.** [thm:momentum-under-translation] *Applying $\hat{T}(a)$ to the momentum operator $\hat{p}$ (see remark 1.1) gives*

$$\hat{T}(a)^\dagger\,\hat{p}\,\hat{T}(a) \;=\; \hat{p}.$$

Proof. Define $\hat{p}' := \hat{T}(a)^\dagger\,\hat{p}\,\hat{T}(a)$, and consider its action on a wavefunction $\psi(x)$:

$$\hat{p}'\psi(x) = \hat{T}(a)^\dagger\,\hat{p}\,\hat{T}(a)\,\psi(x) = \hat{T}(a)^\dagger\,\hat{p}\,\psi(x-a)$$

According to the definition of definition momentum operator position, $\hat{p} = -i\hbar d/dx$, plug in:

$$\hat{p}'\psi(x) = -\hat{T}(a)^\dagger\,i\hbar\,\frac{d\psi(x-a)}{dx}$$

Use $\hat{T}(a)^\dagger = \hat{T}(-a)$,

$$\hat{p}'\psi(x) = -\hat{T}(-a)\,i\hbar\,\frac{d\psi(x-a)}{dx} = -i\hbar\,\frac{d\psi(x)}{dx} = \hat{p}\psi(x)$$

So $\hat{p}'\,\psi = \hat{p}\,\psi$ for every ψ. Since two operators agreeing on all inputs are equal, we conclude

$$\hat{p}' = \hat{p}.$$

□

Theorem 3.6. *For all operator, which only consists of $\hat{x}$ and $\hat{p}$, the conjugation action of $\hat{T}(a)$ on the operator is equivalent to replacing every $\hat{x}$ with $\hat{x} + a\hat{I}$ and leaving every $\hat{p}$ unchanged. ie,*

$$Q(\hat{x},\hat{p}) \longmapsto \hat{T}(a)^\dagger Q(\hat{x},\hat{p})\hat{T}(a) = Q(\hat{x} + a\hat{I},\hat{p})$$

Theorem 3.7 (Translation Symmetry Condition on the Potential)**.** [thm:translation-condition-potential] *Let $\hat{H}$ be a 1D Hamiltonian. If $\hat{T}(a)$ is a symmetry operator of $\hat{H}$, then the potential is invariant under translation by a:*

$$V(x+a) = V(x) \qquad \textit{for all } x.$$

Proof. Since $\hat{T}(a)$ is a unitary symmetry, corollary 1.1 gives

$$\hat{T}(a)^\dagger \hat{H}\hat{T}(a) = \hat{H} \iff \mathrm{comm}\Big(\hat{H},\,\hat{T}(a)\Big) = 0.$$

Notice that $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, and by theorem 3.5 and theorem 3.4, we have

$$\hat{T}(a)^\dagger \hat{H}\hat{T}(a) = \hat{T}(a)^\dagger \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)\hat{T}(a) = \frac{\hat{p}^2}{2m} + V(\hat{x} + a\hat{I}).$$

Which means, $\hat{T}(a)$ is a symmetry of $\hat{H}$ iff

$$V(x) = V(x+a)$$

□

Theorem 3.8 (Eigenvalues of the Translation Operator)**.** [thm:translation-eigenvalues] *The eigenvalues of the translation operator $\hat{T}(a)$ have modulus 1:*

$$\hat{T}(a)\,|\psi\rangle = \lambda\,|\psi\rangle \implies \lambda = e^{i\theta} \quad \textit{for some } \theta \in [0, 2\pi).$$

Proof. Since $\hat{T}(a)$ is unitary (theorem 3.3),

$$\langle \hat{T}(a)\,\psi \mid \hat{T}(a)\,\psi\rangle = \langle \psi \mid \hat{T}(a)^\dagger \hat{T}(a) \mid \psi\rangle = \langle \psi \mid \psi\rangle\,.$$

On the other hand,

$$\langle \hat{T}(a)\,\psi \mid \hat{T}(a)\,\psi\rangle = \langle \lambda\psi \mid \lambda\psi\rangle = |\lambda|^2\,\langle \psi \mid \psi\rangle\,.$$

Equating and dividing by $\langle \psi \mid \psi\rangle \neq 0$:

$$|\lambda|^2 = 1 \implies \lambda = e^{i\theta} \quad \text{for some } \theta \in [0, 2\pi).$$

□

Theorem 3.9 (Bloch's Theorem)**.** [thm:bloch-theorem] *If $T(a)$ is a symmetry of $\hat{H}$ for some $a \in \mathbb{R}$, stationary states of $\hat{H}$ have the form*

$$\psi(x) = e^{iqx} u(x)$$

Where

- *q is called the crystal momentum*
- *$u(x)$ is periodic and have a period of a*

Remark 3.2 (Stationary States may not be Periodic)*.* [rem:stationary-states-may-not-be-periodic] From theorem 3.9, we can see that the stationary states $\psi(x)$ is not periodic. However $u(x)$ is periodic with period a.

Proof. Since $\hat{T}(a)$ is a symmetry of $\hat{H}$, we have $\operatorname{comm}\left(\hat{H}, \hat{T}(a)\right) = 0$. According to theorem commuting share eigenvectors, stationary states of $\hat{H}$ can be chosen as simultaneous eigenvectors of $\hat{T}(a)$: $\hat{T}(a)\,|\psi\rangle = \lambda\,|\psi\rangle$.
By theorem 3.8, $\lambda = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Since $\hat{T}(a)$ may be written as $e^{-iap/\hbar}$, we identify $\theta = qa$ where $q = -p/\hbar$.
Then, we have

$$\hat{T}(a)\,|\psi\rangle = e^{iqa}\,|\psi\rangle \implies \psi(x-a) = e^{iqa}\psi(x)$$

Change $a \to -a$ to rewrite in the standard form:

$$\psi(x+a) = e^{iqa}\,\psi(x).$$

Define $u(x) := e^{-iqx}\,\psi(x)$. Then

$$u(x+a) = e^{-iq(x+a)}\,\psi(x+a) = e^{-iqx}\,e^{-iqa}\,e^{iqa}\,\psi(x) = u(x),$$

where $e^{-iqa}\,e^{iqa} = 1$. So u has period a, and rearranging gives $\psi(x) = e^{iqx}\,u(x)$.
Although $\hat{H}$ is translation-invariant, the state ψ is *not*:

$$\psi(x+a) = e^{iqa}\,\psi(x) \neq \psi(x).$$

However, the observable $|\psi|^2$ still respects the symmetry, since $|e^{iqa}| = 1$:

$$|\psi(x+a)|^2 = |e^{iqa}|^2\,|\psi(x)|^2 = |\psi(x)|^2.$$

□

Theorem 3.10 (Continuous Translation Symmetry Forces Constant V)**.** [thm:continuous-translation-constant-v] *I $\hat{T}(a)$ is a symmetry of $\hat{H}$ for every $a \in \mathbb{R}$, then V is constant.*

TODO

4 Rotational Symmetry in 3D

Definition 4.1 (Rotation Operator)**.** [def:rotation-operator] The *rotation operator* $\hat{R}_{\hat{z}}(\varphi)\psi$ rotates a state ψ about the z-axis by an angle φ:

$$\hat{R}_{\hat{z}}(\varphi)\psi(r,\theta,\phi) = \psi(r,\theta,\phi-\varphi)$$

Remark 4.1. We may also define rotation operators about the x-axis and y-axis, $\hat{R}_{\hat{x}}(\varphi)$ and $\hat{R}_{\hat{y}}(\varphi)$, in a similar manner.

Theorem 4.1 (Exponential Form of the Rotational Operator)**.** [thm:rotation-exponential] *The rotation operator about the z-axis can be written as:*

$$\hat{R}_z(\varphi) = \exp\left[-\frac{i\varphi}{\hbar}\hat{L}_z\right]$$

Proof. Do taylor's expansion on the exponential function:

$$\hat{R}_z(\varphi) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\varphi}{\hbar}\hat{L}_z\right)^n$$

According to definition angular momentum, the angular momentum operator about the z-axis is defined as

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

.
Which means that $\hat{R}_z(\varphi)\psi(r,\theta,\phi)$ is

$$\hat{R}_z(\varphi)\psi(r,\theta,\phi) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\varphi}{\hbar}\left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right)\right)^n\psi(r,\theta,\phi)$$

Plug in the definition of $\hat{p}_x$ and $\hat{p}_y$ from definition momentum operator position, we have

$$\begin{aligned}\hat{R}_z(\varphi)\psi(r,\theta,\phi) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\varphi}{\hbar}\left(\hat{x}(-i\hbar\tfrac{\partial}{\partial y}) - \hat{y}(-i\hbar\tfrac{\partial}{\partial x})\right)\right)^n\psi(r,\theta,\phi)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\varphi\left(\hat{x}\tfrac{\partial}{\partial y} - \hat{y}\tfrac{\partial}{\partial x}\right)\right)^n\psi(r,\theta,\phi).\end{aligned}$$

In cylindrical/spherical coordinates the chain rule gives $\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x} = \frac{\partial}{\partial\phi}$ (multivariable calculus prerequisite, TODO: prove separately), so

$$\hat{R}_z(\varphi)\psi(r,\theta,\phi) = \sum_{n=0}^{\infty}\frac{(-\varphi)^n}{n!}\frac{\partial^n\psi}{\partial\phi^n}(r,\theta,\phi).$$

This is the Taylor expansion of $\psi(r,\theta,\phi-\varphi)$ around ϕ:

$$\psi(r,\theta,\phi-\varphi) = \sum_{n=0}^{\infty}\frac{(-\varphi)^n}{n!}\frac{\partial^n\psi}{\partial\phi^n}(r,\theta,\phi).$$

Therefore $\hat{R}_z(\varphi)\,\psi(r,\theta,\phi) = \psi(r,\theta,\phi-\varphi)$, matching definition 4.1. □

5 Parity/Inverse Symmetry

Definition 5.1 (Parity Operator). [def:parity-operator] The *parity operator* Π (also called the *inversion operator*) is defined by

$$\Pi\,|\psi(\vec{r})\rangle = |\psi(-\vec{r})\rangle$$

Theorem 5.1 (Parity Squared is Identity). [thm:parity-squared] $\Pi^2 = \hat{I}$.

Proof. Applying Π twice to any ψ:

$$\Pi^2\,\psi(\vec{r}) = \Pi\,\psi(-\vec{r}) = \psi(-(-\vec{r})) = \psi(\vec{r}).$$

Since this holds for every ψ, $\Pi^2 = \hat{I}$. □

Theorem 5.2 (Eigenvalues of the Parity Operator). [thm:parity-eigenvalues] *The eigenvalues of the parity operator* Π *are* ± 1.

Proof. If $\Pi\,|\lambda\rangle = \lambda\,|\lambda\rangle$, then $\Pi^2\,|\lambda\rangle = \lambda^2\,|\lambda\rangle$. By theorem 5.1, $\Pi^2 = \hat{I}$, so $\lambda^2 = 1$, hence $\lambda = \pm 1$. □

Theorem 5.3 (Parity Operator is Hermitian). [thm:parity-hermitian] *The parity operator* Π *is Hermitian, i.e.* $\Pi^\dagger = \Pi$.

Proof. For any state $|f\rangle$, $|g\rangle$, we have

$$\langle f \mid \Pi g\rangle = \int_{-\infty}^{\infty} f^*(x)\Pi g(x)\,dx = \int_{-\infty}^{\infty} f^*(x)g(-x)\,dx = \int_{-\infty}^{\infty} (f(-x))^* g(x)\,dx = \langle \Pi f \mid g\rangle$$

Notice that $\langle f \mid \Pi g\rangle = \langle \Pi f \mid g\rangle \implies \Pi = \Pi^\dagger$. So Π is Hermitian. □

Theorem 5.4 (Parity Operator is Unitary). [thm:parity-unitary] *The parity operator* Π *is unitary, i.e.* $\Pi^\dagger = \Pi^{-1}$.

Proof. Since $\Pi^\dagger = \Pi$, and according to theorem 5.1, $\Pi\Pi = \Pi\Pi^\dagger = \hat{I}$, so $\Pi^\dagger = \Pi^{-1}$, and Π is unitary. □

Theorem 5.5 (Parity Operator is Discrete Symmetry). [thm:parity-discrete] *The parity operator* Π *is a discrete symmetry.*

Proof. The parity operators form a group:

- Set $G := \{\hat{I}, \Pi\}$.
- Operation $\cdot$: operator composition.
- Identity $e := \hat{I}$.
- Inverse $\Pi^{-1} := \Pi$ (by theorem 5.1, parity is self-inverse).

This is a discrete group with two elements, so parity is a definition 2.5. □

Theorem 5.6 (Position Operator After Applying Parity). [thm:position-under-parity] *Applying* Π *to the position operator* $\hat{x}$ *(see remark 1.1) gives*

$$\Pi^\dagger\,\hat{x}\,\Pi = -\hat{x}.$$

Proof. Consider $\Pi^\dagger\hat{x}\Pi$ acting on a test function $f(x)$:

$$\Pi^\dagger\hat{x}\Pi\,f(x) = \Pi^\dagger\hat{x}\,f(-x) = \Pi^\dagger\big(x\,f(-x)\big).$$

Since Π is hermitian and unitary, we have $\Pi^\dagger = \Pi$. Define $g(x) := x\,f(-x)$; then

$$\Pi^\dagger\hat{x}\Pi\,f(x) = \Pi\,g(x) = g(-x) = -x\,f(x).$$

So $\Pi^\dagger\hat{x}\Pi\,f = -\hat{x}\,f$ for every f, giving $\Pi^\dagger\hat{x}\Pi = -\hat{x}$. □

Theorem 5.7 (Momentum Operator After Applying Parity). [thm:momentum-under-parity] *Applying* Π *to the momentum operator* $\hat{p}$ *gives*

$$\Pi^\dagger \hat{p}\, \Pi = -\hat{p}.$$

Proof. Consider $\Pi^\dagger \hat{p} \Pi$ acting on a test function $f(x)$:

$$\Pi^\dagger \hat{p} \Pi\, f(x) = \Pi^\dagger \hat{p}\, f(-x) = \Pi^\dagger \left(-i\hbar \frac{d}{dx} f(-x) \right).$$

Since Π is hermitian and unitary, we have $\Pi^\dagger = \Pi$. Define $g(x) := -i\hbar \frac{d}{dx} f(-x)$; then

$$\Pi^\dagger \hat{p} \Pi\, f(x) = \Pi\, g(x) = g(-x) = i\hbar \frac{d}{dx} f(x).$$

So $\Pi^\dagger \hat{p} \Pi\, f = -\hat{p}\, f$ for every f, giving $\Pi^\dagger \hat{p} \Pi = -\hat{p}$. □

Theorem 5.8 (General Operator After Applying Parity). [thm:operator-under-parity] *For any operator* $Q(\hat{x}, \hat{p})$ *built from* $\hat{x}$ *and* $\hat{p}$,

$$\Pi^\dagger\, Q(\hat{x}, \hat{p})\, \Pi = Q(-\hat{x}, -\hat{p}).$$

Theorem 5.9 (Angular Momentum is Parity-Invariant). [thm:angular-momentum-under-parity] $\Pi^\dagger\, \hat{\vec{L}}\, \Pi = \hat{\vec{L}}$.

Proof. Apply theorem 5.8 to $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$:

$$\Pi^\dagger\, (\hat{\vec{r}} \times \hat{\vec{p}})\, \Pi = (-\hat{\vec{r}}) \times (-\hat{\vec{p}}) = \hat{\vec{r}} \times \hat{\vec{p}} = \hat{\vec{L}}.$$

□

Theorem 5.10 (Parity of Atomic Eigenstates). [thm:parity-atomic-eigenstates] *For an definition atomic eigenstate* $|\psi_{n\ell m}\rangle$,

$$\Pi\, |\psi_{n\ell m}\rangle = (-1)^\ell\, |\psi_{n\ell m}\rangle\,.$$

Proof. **TODO** □

Theorem 5.11 (Laporte's Rule). [thm:laporte-rule] *For atomic eigenstates, the electric-dipole matrix element vanishes when* $\ell' + \ell$ *is even:*

$$\langle n'\ell'm' | \vec{r} | n\ell m \rangle = 0 \quad \text{if } \ell' + \ell \text{ is even.}$$

Proof. **TODO** □

6 Time-Translation Symmetry

Definition 6.1 (Time-Translation Operator). [def:time-translation-operator] The *time-translation operator* $U(t)$ is defined by

$$U(t)\, \Psi(x, a) = \Psi(x, a + t).$$

Theorem 6.1 (Exponential Form of the Time-Translation Operator). [thm:time-translation-exponential] *The time-translation operator can be written as*

$$U(t) = e^{it\hat{H}/\hbar},$$

where we assume the Hamiltonian $\hat{H}$ *is independent of time.*

Proof. **TODO** □

Theorem 6.2 (Hermitian Conjugate of $U(t)$)**.** [thm:time-translation-hermitian-conjugate] *The Hermitian conjugate of the time-translation operator is*

$$U^\dagger(t) = U(-t).$$

Proof. **TODO** □

Theorem 6.3 (Time Derivative of $U(t)$)**.** [thm:time-translation-derivative] *The time derivative of the time-translation operator satisfies*

$$\frac{d}{dt}U(t) = \frac{i\hat{H}}{\hbar}U(t).$$

Proof. **TODO** □

Theorem 6.4 ($U(t)$ Commutes with $\hat{H}$)**.** [thm:time-translation-commutes-with-h] *The time-translation operator commutes with the Hamiltonian:*

$$\operatorname{comm}\left(U(t),\, \hat{H}\right) = 0.$$

Proof. **TODO** □

7 Pictures

Definition 7.1 (Pictures)**.** [def:pictures] In quantum mechanics, a *picture* is a convention for how to distribute the time dependence between states and operators

Definition 7.2 (Schrödinger Picture)**.** [def:schrodinger-picture] The *Schrödinger picture* keeps the state dependent on time, while the operators are time-independent. ie,

$$\text{state: } |S(t)\rangle \text{ , expected value: } \langle Q\rangle = \langle S(t)|Q|S(t)\rangle$$

Definition 7.3 (Heisenberg Picture)**.** [def:heisenberg-picture] The *Heisenberg picture* keeps the state independent of time. Instead, the time dependence is transferred to the operators.

$$\text{state: } |H\rangle \text{ , expected value: } \langle Q_H(t)\rangle = \langle H|Q_H(t)|H\rangle$$

where $U(t)$ is the time-translation operator.

Remark 7.1 (Pictures Are Conjugation Actions). [rem:pictures-are-conjugation] The Heisenberg operator $Q_H(t) = U^\dagger(t)\, Q\, U(t)$ is the conjugation action of $U(t)$ on Q. Expectation values agree between the Schrödinger and Heisenberg pictures because that is exactly what the conjugation action preserves.

Example 7.1. Consider $H = \frac{p^2}{2m} + V(x)$, find in the Heisenberg picture, the $x_H\delta$ operator where δ is infinitesimal.
Since δ is small, we may do taylor's expansion and only keep the first order term (at t = 0), and we know that $e^{it\hat{H}}\hbar' = \frac{i\hat{H}}{\hbar}$

$$x_H(\delta) = \hat{U}^\dagger(\delta)\hat{x}\hat{U}(\delta) \approx \left(1 + \frac{i}{\hbar}\delta\hat{H}\right)\hat{x}\left(1 - \frac{i}{\hbar}\delta\hat{H}\right)$$

Expand and omitting the second order term, we have

$$x_H(\delta) \approx \hat{x} + \frac{i}{\hbar}\operatorname{comm}\left(\hat{H},\, \hat{x}\right)\delta$$

Apply Hamiltonian Commutators to substitute $\operatorname{comm}\left(\hat{H},\, \hat{x}\right) = -\dfrac{i\hbar}{m}\hat{p}$:

$$x_H(\delta) \approx \hat{x} + \frac{i}{\hbar} \cdot \left(-\frac{i\hbar}{m}\hat{p}\right) \delta = \hat{x} + \frac{\hat{p}}{m}\,\delta.$$

Remark 7.2. This matches the classical equation of motion $v = p/m$: the infinitesimal time-evolution of $\hat{x}$ in the Heisenberg picture reproduces the classical relation of $x = x_0 + vt$.

Lemma 7.1 (Conjugation Preserves Commutator with a Commuting Operator)**.** [lem:conjugation-preserves-commu] *If* $\operatorname{comm}\left(\hat{A},\, \hat{C}\right) = 0$*, then for any operator* $\hat{X}$*,*

$$\hat{A}^\dagger \operatorname{comm}\left(\hat{X},\, \hat{C}\right) \hat{A} \;=\; \operatorname{comm}\left(\hat{A}^\dagger \hat{X} \hat{A},\, \hat{C}\right).$$

Proof. **TODO** □

Theorem 7.1 (Heisenberg Equation of Motion)**.** [thm:heisenberg-equation] *Assuming* $\hat{Q}$ *and* $\hat{H}$ *are independent of time*

$$\frac{d}{dt}\hat{Q}_H(t) \;=\; \frac{1}{i\hbar} \operatorname{comm}\left(\hat{Q}_H(t),\, \hat{H}\right).$$

Proof. By the definition of the Heisenberg picture, $\hat{Q}_H(t) = U^\dagger(t)\, \hat{Q}\, U(t)$. Take the total time derivative using the product rule:

$$\frac{d\hat{Q}_H}{dt} = \frac{dU^\dagger}{dt}\, \hat{Q}\, U + U^\dagger\, \frac{d\hat{Q}}{dt}\, U + U^\dagger\, \hat{Q}\, \frac{dU}{dt}.$$

Since $\hat{Q}$ is time-independent, $d\hat{Q}/dt = 0$. From theorem 6.3, $\dot{U}(t) = \frac{i\hat{H}}{\hbar} U(t)$, and taking the adjoint, $\dot{U}^\dagger(t) = -\frac{i\hat{H}}{\hbar} U^\dagger(t)$. Substituting:

$$\frac{d\hat{Q}_H}{dt} = -\frac{i\hat{H}}{\hbar}\, U^\dagger\, \hat{Q}\, U + U^\dagger\, \hat{Q}\, \frac{i\hat{H}}{\hbar}\, U.$$

From theorem 6.4, $\operatorname{comm}\left(U(t),\, \hat{H}\right) = 0$, so $\hat{H}$ passes through both U and $U^\dagger$ freely. Rearranging:

$$\frac{d\hat{Q}_H}{dt} = -\frac{i}{\hbar}\, \hat{H}\, (U^\dagger\, \hat{Q}\, U) + \frac{i}{\hbar}\, (U^\dagger\, \hat{Q}\, U)\, \hat{H}.$$

Recognize $U^\dagger\, \hat{Q}\, U = \hat{Q}_H$:

$$\frac{d\hat{Q}_H}{dt} = -\frac{i}{\hbar}\, \hat{H}\, \hat{Q}_H + \frac{i}{\hbar}\, \hat{Q}_H\, \hat{H} = \frac{i}{\hbar} \operatorname{comm}\left(\hat{Q}_H,\, \hat{H}\right) = \frac{1}{i\hbar} \operatorname{comm}\left(\hat{H},\, \hat{Q}_H\right).$$

□

Corollary 7.1 (Energy Conservation in Heisenberg Picture)**.** [cor:energy-conservation-heisenberg] *If* $\hat{H}$ *is independent of time, then* $\frac{d\hat{H}}{dt} = 0$*.*

Proof. **TODO** □