Differentiation and Integration Cheatsheet

Function notation: $\mathrm{D}^n(f)$ and $\mathrm{D}^{-1}(f)$

Reading the sheet. Functions have a real input and real or complex values. Each $x \mapsto \cdots$ is a whole function. $\mathrm{D}(f)$ is its derivative function; $\mathrm{D}^0(f) = f$ and $\mathrm{D}^{n+1}(f) = \mathrm{D}(\mathrm{D}^n(f))$. Formal definitions of differentiation and antiderivatives, real-analysis are still owed.

Constants and endpoint evaluation. The left column gives one primitive $F = \mathrm{D}^{-1}(f)$, with $\mathrm{D}(F) = f$. All primitives on an interval form the family

$$\mathcal{F} \equiv C \mapsto (x \mapsto F(x) + C)\,.$$

The endpoint notation means $[F]_a^b \equiv F(b) - F(a)$; for this family, $[\mathcal{F}]_a^b \equiv F(b) - F(a)$, since every member has that difference. Primitive identities below hold up to an additive constant. At infinite endpoints, use the corresponding limits, when they exist.

Pointwise operations. For example, $f \cdot g \equiv x \mapsto f(x) \cdot g(x)$ and $f \circ g \equiv x \mapsto f(g(x))$. All formulas are used on intervals where the functions and indicated operations are defined.

Basic rules

Here a, b are constants. In the quotient rule, $g(x) \neq 0$.

$$\begin{aligned}
\mathrm{D}(a \cdot f + b \cdot g) &= a \cdot \mathrm{D}(f) + b \cdot \mathrm{D}(g), \\
\mathrm{D}(f \cdot g) &= \mathrm{D}(f) \cdot g + f \cdot \mathrm{D}(g), \\
\mathrm{D}(f/g) &= \frac{\mathrm{D}(f) \cdot g - f \cdot \mathrm{D}(g)}{g^2}, \\
\mathrm{D}(f \circ g) &= (\mathrm{D}(f) \circ g) \cdot \mathrm{D}(g), \\
\mathrm{D}^{-1}(a \cdot f + b \cdot g) &= a \cdot \mathrm{D}^{-1}(f) + b \cdot \mathrm{D}^{-1}(g), \\
\mathrm{D}^{-1}(f \cdot \mathrm{D}(g)) &= f \cdot g - \mathrm{D}^{-1}(\mathrm{D}(f) \cdot g) && \text{(integration by parts)}, \\
\mathrm{D}^{-1}((f \circ g) \cdot \mathrm{D}(g)) &= \mathrm{D}^{-1}(f) \circ g && \text{(substitution)}.
\end{aligned}$$

Power, exponential and logarithmic functions

For real p, use $x > 0$ in the power row; integer powers extend wherever defined. For logarithms use $x \neq 0$; $k \neq 0$, $b > 0$, and $b \neq 1$ where it is a logarithm base.

$\mathrm{D}^{-1}(f)$	f	$\mathrm{D}(f)$
$x \mapsto c \cdot x$	$x \mapsto c$	$x \mapsto 0$
$x \mapsto \dfrac{x^{p+1}}{p+1}$	$x \mapsto x^p \quad (p \neq -1)$	$x \mapsto p \cdot x^{p-1}$
$x \mapsto \ln \lvert x \rvert$	$x \mapsto \dfrac{1}{x}$	$x \mapsto -\dfrac{1}{x^2}$
$x \mapsto \dfrac{e^{k \cdot x}}{k}$	$x \mapsto e^{k \cdot x}$	$x \mapsto k \cdot e^{k \cdot x}$
$x \mapsto \dfrac{b^x}{\ln b}$	$x \mapsto b^x$	$x \mapsto \ln b \cdot b^x$
$x \mapsto x \cdot \ln \lvert x \rvert - x$	$x \mapsto \ln \lvert x \rvert$	$x \mapsto \dfrac{1}{x}$
$x \mapsto x \cdot \ln \lvert k \cdot x \rvert - x$	$x \mapsto \ln \lvert k \cdot x \rvert$	$x \mapsto \dfrac{1}{x}$
$x \mapsto \dfrac{x \cdot \ln x - x}{\ln b}$	$x \mapsto \log_b x \quad (x > 0)$	$x \mapsto \dfrac{1}{x \cdot \ln b}$

Trigonometric functions

Throughout this page, $k \in \mathbb{R} \setminus \{0\}$. Avoid poles and logarithm zeros.

$\mathrm{D}^{-1}(f)$	f	$\mathrm{D}(f)$
$x \mapsto -\frac{\cos(k \cdot x)}{k}$	$x \mapsto \sin(k \cdot x)$	$x \mapsto k \cdot \cos(k \cdot x)$
$x \mapsto \frac{\sin(k \cdot x)}{k}$	$x \mapsto \cos(k \cdot x)$	$x \mapsto -k \cdot \sin(k \cdot x)$
$x \mapsto -\frac{\ln\lvert\cos(k \cdot x)\rvert}{k}$	$x \mapsto \tan(k \cdot x)$	$x \mapsto k \cdot \sec^2(k \cdot x)$
$x \mapsto \frac{\ln\lvert\sin(k \cdot x)\rvert}{k}$	$x \mapsto \cot(k \cdot x)$	$x \mapsto -k \cdot \csc^2(k \cdot x)$
$x \mapsto \frac{\ln\lvert\sec(k \cdot x) + \tan(k \cdot x)\rvert}{k}$	$x \mapsto \sec(k \cdot x)$	$x \mapsto k \cdot \sec(k \cdot x) \cdot \tan(k \cdot x)$
$x \mapsto -\frac{\ln\lvert\csc(k \cdot x) + \cot(k \cdot x)\rvert}{k}$	$x \mapsto \csc(k \cdot x)$	$x \mapsto -k \cdot \csc(k \cdot x) \cdot \cot(k \cdot x)$

Hyperbolic functions

$\mathrm{D}^{-1}(f)$	f	$\mathrm{D}(f)$
$x \mapsto \frac{\cosh(k \cdot x)}{k}$	$x \mapsto \sinh(k \cdot x)$	$x \mapsto k \cdot \cosh(k \cdot x)$
$x \mapsto \frac{\sinh(k \cdot x)}{k}$	$x \mapsto \cosh(k \cdot x)$	$x \mapsto k \cdot \sinh(k \cdot x)$
$x \mapsto \frac{\ln(\cosh(k \cdot x))}{k}$	$x \mapsto \tanh(k \cdot x)$	$x \mapsto \frac{k}{\cosh^2(k \cdot x)}$
$x \mapsto \frac{\ln\lvert\sinh(k \cdot x)\rvert}{k}$	$x \mapsto \coth(k \cdot x)$	$x \mapsto -k \cdot \operatorname{csch}^2(k \cdot x)$

Products and squares

$\mathrm{D}^{-1}(f)$	f
$x \mapsto \frac{\tan(k \cdot x)}{k}$	$x \mapsto \sec^2(k \cdot x)$
$x \mapsto -\frac{\cot(k \cdot x)}{k}$	$x \mapsto \csc^2(k \cdot x)$
$x \mapsto \frac{\sec(k \cdot x)}{k}$	$x \mapsto \sec(k \cdot x) \cdot \tan(k \cdot x)$
$x \mapsto -\frac{\csc(k \cdot x)}{k}$	$x \mapsto \csc(k \cdot x) \cdot \cot(k \cdot x)$
$x \mapsto \frac{x}{2} - \frac{\sin(2k \cdot x)}{4k}$	$x \mapsto \sin^2(k \cdot x)$
$x \mapsto \frac{x}{2} + \frac{\sin(2k \cdot x)}{4k}$	$x \mapsto \cos^2(k \cdot x)$

Inverse trigonometric functions

Use $|x| < 1$ for arcsin and arccos. Here $\operatorname{arccot} x \equiv \pi/2 - \arctan x$; $\operatorname{arcsec} x \equiv \arccos(1/x)$ and $\operatorname{arccsc} x \equiv \arcsin(1/x)$ are listed for $x > 1$.

$\mathrm{D}^{-1}(f)$	f	$\mathrm{D}(f)$
$x \mapsto x \cdot \arcsin x + \sqrt{1-x^2}$	$x \mapsto \arcsin x$	$x \mapsto \dfrac{1}{\sqrt{1-x^2}}$
$x \mapsto x \cdot \arccos x - \sqrt{1-x^2}$	$x \mapsto \arccos x$	$x \mapsto -\dfrac{1}{\sqrt{1-x^2}}$
$x \mapsto x \cdot \arctan x - \dfrac{\ln(1+x^2)}{2}$	$x \mapsto \arctan x$	$x \mapsto \dfrac{1}{1+x^2}$
$x \mapsto x \cdot \operatorname{arccot} x + \dfrac{\ln(1+x^2)}{2}$	$x \mapsto \operatorname{arccot} x$	$x \mapsto -\dfrac{1}{1+x^2}$
$x \mapsto x \cdot \operatorname{arcsec} x - \ln(x+\sqrt{x^2-1})$	$x \mapsto \operatorname{arcsec} x$	$x \mapsto \dfrac{1}{x \cdot \sqrt{x^2-1}}$
$x \mapsto x \cdot \operatorname{arccsc} x + \ln(x+\sqrt{x^2-1})$	$x \mapsto \operatorname{arccsc} x$	$x \mapsto -\dfrac{1}{x \cdot \sqrt{x^2-1}}$

Quadratic expressions

Here $a > 0$. Use $|x| < a$ for $\sqrt{a^2-x^2}$, $|x| > a$ for $\sqrt{x^2-a^2}$, and $x \neq \pm a$ for $1/(x^2-a^2)$.

$\mathrm{D}^{-1}(f)$	f
$x \mapsto \dfrac{1}{a} \cdot \arctan\left(\dfrac{x}{a}\right)$	$x \mapsto \dfrac{1}{x^2+a^2}$
$x \mapsto \dfrac{1}{2a} \cdot \ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$	$x \mapsto \dfrac{1}{x^2-a^2}$
$x \mapsto \arcsin\left(\dfrac{x}{a}\right)$	$x \mapsto \dfrac{1}{\sqrt{a^2-x^2}}$
$x \mapsto \ln\lvert x+\sqrt{x^2+a^2}\rvert$	$x \mapsto \dfrac{1}{\sqrt{x^2+a^2}}$
$x \mapsto \ln\lvert x+\sqrt{x^2-a^2}\rvert$	$x \mapsto \dfrac{1}{\sqrt{x^2-a^2}}$
$x \mapsto \dfrac{x}{a^2 \cdot \sqrt{x^2+a^2}}$	$x \mapsto \dfrac{1}{(x^2+a^2)^{3/2}}$
$x \mapsto \dfrac{x}{2} \cdot \sqrt{a^2-x^2} + \dfrac{a^2}{2} \cdot \arcsin\left(\dfrac{x}{a}\right)$	$x \mapsto \sqrt{a^2-x^2}$
$x \mapsto \dfrac{x}{2} \cdot \sqrt{x^2+a^2} + \dfrac{a^2}{2} \cdot \ln\lvert x+\sqrt{x^2+a^2}\rvert$	$x \mapsto \sqrt{x^2+a^2}$

Substitution as a function

Choose $x = \phi(\theta)$. Apply the substitution rule to $(f \circ \phi) \cdot \mathrm{D}(\phi)$, including the derivative factor. For $a > 0$:

Expression in x	ϕ	$\mathrm{D}(\phi)$
$\sqrt{a^2 - x^2}$	$\theta \mapsto a \cdot \sin\theta$	$\theta \mapsto a \cdot \cos\theta$
$\sqrt{x^2 + a^2}$	$\theta \mapsto a \cdot \tan\theta$	$\theta \mapsto a \cdot \sec^2\theta$
$\sqrt{x^2 - a^2}$	$\theta \mapsto a \cdot \sec\theta$	$\theta \mapsto a \cdot \sec\theta \cdot \tan\theta$

The first two substitutions use $-\pi/2 < \theta < \pi/2$; the third uses $0 < \theta < \pi/2$ and covers $x > a$. The respective square roots become $a \cdot \cos\theta$, $a \cdot \sec\theta$, and $a \cdot \tan\theta$.
For a linear substitution $\phi \equiv x \mapsto k \cdot x + q$, with $k \neq 0$,

$$\mathrm{D}^{-1}(f \circ \phi) = \frac{1}{k} \cdot \left(\mathrm{D}^{-1}(f) \circ \phi\right),$$
$$\left[\mathrm{D}^{-1}(f \circ \phi)\right]_a^b = \frac{1}{k} \cdot \left[\mathrm{D}^{-1}(f)\right]_{\phi(a)}^{\phi(b)}.$$

Product-to-sum identities

$$\sin A \cdot \sin B = \tfrac{1}{2}\left(\cos(A - B) - \cos(A + B)\right),$$
$$\cos A \cdot \cos B = \tfrac{1}{2}\left(\cos(A - B) + \cos(A + B)\right),$$
$$\sin A \cdot \cos B = \tfrac{1}{2}\left(\sin(A + B) + \sin(A - B)\right).$$

For $\alpha \neq \beta$ and $\alpha \neq -\beta$,

$$\mathrm{D}^{-1}\left(x \mapsto \sin(\alpha \cdot x) \cdot \sin(\beta \cdot x)\right) = x \mapsto \frac{\sin((\alpha - \beta) \cdot x)}{2(\alpha - \beta)} - \frac{\sin((\alpha + \beta) \cdot x)}{2(\alpha + \beta)}.$$

For equal frequencies, use the $\sin^2(k \cdot x)$ row instead.

Gaussian formulas and symmetric intervals

For $a > 0$, $b \in \mathbb{R}$ and $n \in \mathbb{N}_0$,

$$\left[\mathrm{D}^{-1}\left(x \mapsto e^{-a \cdot x^2}\right)\right]_{-\infty}^{\infty} = \sqrt{\frac{\pi}{a}},$$
$$\left[\mathrm{D}^{-1}\left(x \mapsto x^2 \cdot e^{-a \cdot x^2}\right)\right]_{-\infty}^{\infty} = \frac{1}{2a}\sqrt{\frac{\pi}{a}},$$
$$\left[\mathrm{D}^{-1}\left(x \mapsto x^{2n+1} \cdot e^{-a \cdot x^2}\right)\right]_{-\infty}^{\infty} = 0,$$
$$\left[\mathrm{D}^{-1}\left(x \mapsto e^{-a \cdot x^2} \cdot e^{i \cdot b \cdot x}\right)\right]_{-\infty}^{\infty} = \sqrt{\frac{\pi}{a}} \cdot e^{-b^2/(4a)}.$$

For a continuous function on $[-L, L]$ with $L > 0$,

$$\left[\mathrm{D}^{-1}(f)\right]_{-L}^{L} = \begin{cases} 0, & f(-x) = -f(x) \quad \text{(odd)}, \\ 2 \cdot \left[\mathrm{D}^{-1}(f)\right]_0^L, & f(-x) = f(x) \quad \text{(even)}. \end{cases}$$