Chapter 1: Complex Numbers and Euler's Formula

Mathematical Methods

1 Complex Numbers

Definition 1.1 (Imaginary Unit)**.** [def:imaginary-number] An *imaginary unit* is defined as following:

$$i^2 = -1$$

Definition 1.2 (Complex Number)**.** [def:complex-number] The type of *complex numbers* is defined as following:

$$\mathbb{C} \equiv \{a + b \cdot i \mid a, b : \mathbb{R}\}$$

An element $z : \mathbb{C}$ is a *complex number*

`LaTeX: \CC`

For $z : \mathbb{C}$, write $z = a + b \cdot i$, where $a, b : \mathbb{R}$

Definition 1.3 (Real Part)**.** [def:real-part] The real part function is defined as following:

$$\mathrm{Re} : \mathbb{C} \to \mathbb{R} \equiv z \mapsto a$$

`LaTeX: \RealPart (z)`

Definition 1.4 (Imaginary Part)**.** [def:imaginary-part] The imaginary part function is defined as following:

$$\mathrm{Im} : \mathbb{C} \to \mathbb{R} \equiv z \mapsto b$$

`LaTeX: \ImagPart (z)`

Definition 1.5 (Complex Conjugate)**.** [def:complex-conjugate] The complex conjugate function is defined as following:

$$\mathrm{Conj} : \mathbb{C} \to \mathbb{C} \equiv z \mapsto a - b \cdot i$$

`LaTeX: \Conj (z)`

Convention 1.1 (Complex Conjugate Notation)**.** [conv:complex-conjugate-notation] For $z : \mathbb{C}$, we write

$$z^* \equiv \mathrm{Conj}(z)$$

where Conj is the complex conjugate function

`LaTeX: \conj {z}`

2 Arithmetic Operations

Given two complex numbers $z_1 = a + bi$ and $z_2 = c + di$:

Proposition 2.1 (Addition:)**.** [prop:addition]

$$z_1 + z_2 = (a + c) + (b + d)i \tag{1}$$

Proposition 2.2 (Subtraction:)**.** [prop:subtraction]

$$z_1 - z_2 = (a - c) + (b - d)i \tag{2}$$

Proposition 2.3 (Multiplication:). [prop:multiplication]

$$\begin{aligned} z_1 \cdot z_2 &= (a+bi)(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned} \tag{3}$$

Proposition 2.4 (Division:). [prop:division]

$$\frac{z_1}{z_2} = \frac{z_1 \cdot {z_2}^*}{z_2 \cdot {z_2}^*} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \tag{4}$$

3 Euler's Formula

Remark 3.1 (Taylor's Expansion). [rem:trigonometric-taylor-series] Recall from Section 2.3 (Taylor's Expansion), the Taylor series for $\cos\theta$ and $\sin\theta$ around $\theta = 0$:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} \tag{5}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \tag{6}$$

Definition 3.1 (Definition of e). [def:e] The natural number e is defined as:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \approx 2.71828 \tag{7}$$

Proposition 3.1 (Taylor series for e^x). [prop:exponential-taylor-series] *From this, we can derive the Taylor series for e^x by replacing 1 with x in each term:*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \tag{8}$$

Theorem 3.1 (Euler's Formula). [thm:eulers-formula] *This is **Euler's Formula**:*

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{9}$$

Proof. Substitute $x = i\theta$:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \ldots \end{aligned} \tag{10}$$

Separating real and imaginary parts:

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) \\ &= \cos\theta + i\sin\theta \end{aligned} \tag{11}$$

□

4 Polar Form of Complex Numbers

From Section 1.2 (Angles), we know that any point (x, y) in Cartesian coordinates can be expressed in polar coordinates (r, θ) where:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right) \tag{12}$$

Definition 4.1 (Polar Form of Complex Numbers)**.** [def:polar-form] Therefore, a complex number $z = x + iy$ can be written as:

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned} \tag{13}$$

Where $r = |z|$ is the **modulus** and $\theta = \arg(z)$ is the **argument** of the complex number.

Remark 4.1. [rem:polar-form-quadrant] Similar to the transform of a point from Cartesian to Polar coordinates, we must be careful about the quadrant when determining θ from $\arctan\left(\frac{y}{x}\right)$.

5 Multiplication in Polar Form

Proposition 5.1 (Multiplication in Polar Form)**.** [prop:polar-multiplication] *Given two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$:*

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} \tag{14}$$

Multiplication in polar form:

- *Moduli multiply:* $|z_1 z_2| = r_1 \cdot r_2$
- *Arguments add:* $\arg(z_1 z_2) = \theta_1 + \theta_2$

Proposition 5.2. [prop:complex-powers] *For z^n where $z = re^{i\theta}$:*

$$z^n = r^n e^{in\theta} \tag{15}$$

6 Nth Root

To find $w = z^{1/n}$, we need to solve $w^n = z$.

Proposition 6.1 (Nth Root of Complex Numbers)**.** [prop:nth-roots] *This gives exactly n distinct roots:*

$$z^{1/n} = r^{1/n} e^{i\frac{\theta + 2\pi k}{n}}, \quad k = 0, 1, 2, \ldots, n-1 \tag{16}$$

Proof. Let $z = re^{i\theta}$ and $w = \rho e^{i\phi}$. Then:

$$w^n = \rho^n e^{in\phi} = re^{i\theta} \tag{17}$$

Comparing moduli and arguments:

$$\rho^n = r \quad \Rightarrow \quad \rho = r^{1/n} \text{ (unique positive real)} \tag{18}$$

$$n\phi = \theta \quad \Rightarrow \quad \phi = \frac{\theta}{n} \tag{19}$$

However, since angles are periodic with period 2π, we can write $z = re^{i(\theta+2\pi k)}$ for any integer k. Therefore:

$$\phi = \frac{\theta + 2\pi k}{n}, \quad k = 0, 1, 2, \ldots, n-1 \tag{20}$$

□

Remark 6.1. [rem:distinct-nth-roots] Note: When $k = n$, we get $\frac{\theta+2\pi n}{n} = \frac{\theta}{n} + 2\pi$, which is the same angle as $k = 0$. So there are exactly n distinct nth roots.