Tutorial 3

Introduction to Data Analysis Techniques

1 Derivatives and Euler's Method

Example 1.1 (Derivative Review). [eg:difference-formulas-by-hand] Consider

$$f : \mathbb{R} \to \mathbb{R} \equiv x \mapsto 1 + x + x^2 + x^3 + x^4$$

Work at $x = 0$ with step $h = \frac{1}{10}$

1. Find $\mathrm{D}^k(f)(0)$ for $k = 0, \ldots, 4$. Construct the Taylor polynomial through order 4 centred at 0. What is its remainder function?

2. Construct the forward, backward, and central first-derivative approximations $f'_{h,+}$, $f'_{h,-}$, and $f'_{h,0}$, and the central second-derivative approximation $f''_{h,0}$. Give each function's type and rule, using the difference conventions in Lecture 3

3. Calculate all four approximations at 0 for $h = \frac{1}{10}$. Make a table of the approximate value, exact target value, and signed error, using computed minus target

4. Verify that the central first-derivative approximation is the average of the forward and backward approximations. What does the average $\frac{f(h)+f(-h)}{2}$ approximate instead?

Example 1.2 (Second-Order Euler by Hand)**.** [eg:second-order-euler-by-hand] Consider the initial-value problem

$$\forall x \in \mathbb{R}, \quad \mathrm{D}^2(f)(x) = -f(x), \qquad f(0) = 1, \qquad \mathrm{D}(f)(0) = 0$$

Apply forward Euler with $x_0 = 0$, step $h = \frac{1}{2}$, and $N = 2$ steps

1. Introduce $v \equiv \mathrm{D}(f)$ and rewrite the equation as a system of two first-order ODEs. Identify $g(x, y, v)$ in the second-order Euler convention

2. State the types and initial values of $f^\dagger$ and f'. Write their Euler updates. Which values must be used on the right-hand sides at each step?

3. Calculate both steps and complete the table

n	x_n	$f^\dagger(x_n)$	$f'(x_n)$
0	0		
1	$\frac{1}{2}$		
2	1		

4. For comparison, the exact solution is $f(x) = \cos(x)$. Using radians, calculate the signed errors in $f^\dagger(1)$ and $f'(1)$. Use the correct exact target for each

5. Match your two tracked functions and initial values to $U(0)$, $U(1)$, and a in the general-order convention